Diagonal p-permutation functors

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Abstract

Let k be an algebraically closed field of positive characteristic p, and $\mathbb{F}$ be an algebraically closed field of characteristic 0. We consider the $\mathbb{F}$-linear category $\mathbb{F}pp_k^\Delta$ of finite groups, in which the set of morphisms from G to H is the $\mathbb{F}$-linear extension $\mathbb{F}T^\Delta(H,G)$ of the Grothendieck group $T^\Delta(H,G)$ of p-permutation (kH,kG)-bimodules with (twisted) diagonal vertices. The $\mathbb{F}$-linear functors from $\mathbb{F}pp_k^\Delta$ to $\mathbb{F}$-Mod are called *diagonal p-permutation functors.* They form an abelian category $\mathcal{F}_{pp_k}^\Delta$.

We study in particular the functor $\mathbb{F}T^\Delta$ sending a finite group G to the Grothendieck group $\mathbb{F}T(G)$ of p-permutation kG-modules, and show that $\mathbb{F}T^\Delta$ is a semisimple object of $\mathcal{F}_{pp_k}^\Delta$, equal to the direct sum of specific simple functors parametrized by isomorphism classes of pairs (P,s) of a finite p-group P and a generator s of a p'-subgroup acting faithfully on P. This leads to a precise description of the evaluations of these simple functors. In particular, we show that the simple functor indexed by the trivial pair $(1,1)$ is isomorphic to the functor sending a finite group G to $\mathbb{F}K_0(kG)$, where $K_0(kG)$ is the Grothendieck group of projective kG-modules.

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1. Introduction

Let p be a prime number. Throughout we denote by $\mathbb{F}$ an algebraically closed field of characteristic zero, and by k an algebraically closed field of characteristic p. The p-permutation modules play a crucial role in the study of modular representation theory of finite groups. A splendid Rickard equivalence, introduced by Rickard [10], between blocks of finite group algebras is given by a chain complex consisting of p-permutation bimodules. Also a p-permutation equivalence, introduced by Boltje

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and Xu [1], and studied extensively later by Boltje and Perepelitsky [9], is an element in the Grothendieck group of p-permutation bimodules.

In [7], Ducellier studied p-permutation functors: Consider the category $\mathbb{F}pp_k$ where the objects are finite groups and the morphisms between groups G and H are given by the Grothendieck group $\mathbb{F}\otimes_{\mathbb{Z}}T(H,G)$ of p-permutation (kH,kG)-bimodules. A p-permutation functor is an $\mathbb{F}$-linear functor from $\mathbb{F}pp_k$ to $\mathbb{F}$-Mod. The indecomposable direct summands of the bimodules that appears in a p-permutation equivalence between blocks of finite group algebras have twisted diagonal vertices. Therefore, inspired by the work of Ducellier, we consider a category with less morphisms: Let $\mathbb{F}pp_k^{\Delta}$ be a category where the objects are finite groups and the morphisms between groups G and H are given by the Grothendieck group $\mathbb{F}\otimes_{\mathbb{Z}}T^{\Delta}(H,G)$ of p-permutation (kH,kG)-bimodules whose indecomposable direct summands have twisted diagonal vertices. An $\mathbb{F}$-linear functor from $\mathbb{F}pp_k^{\Delta}$ to $\mathbb{F}$-Mod is called a *diagonal p-permutation functor*. Diagonal p-permutation functors form an abelian category $\mathcal{F}_{pp_k}^{\Delta}$.

By [2] and [4], if S is a simple R-linear representation of an R-linear category $\mathcal{C}$ (where R is any commutative ring), and X is any object of $\mathcal{C}$ such that $S(C)\neq\{0\}$, then $S(X)$ is a simple module for the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(X)$ of X in $\mathcal{C}$. Conversely, to any object X of $\mathcal{C}$ and any simple $\mathrm{End}_{\mathcal{C}}(X)$-module V, one can associate a simple R-linear representation $S_{X,V}$ of $\mathcal{C}$, with the property that $S_{X,V}(X)\cong V$. This gives a parametrization of the simple representations of $\mathcal{C}$ by pairs (X,V) of an object X of $\mathcal{C}$ and a simple $\mathrm{End}_{\mathcal{C}}(X)$-module V. However, this parametrization is not one to one in general, as many different pairs (X,V) yield the same simple functor $S_{X,V}$, up to isomorphism.

This applies in particular for the category $\mathcal{C}=\mathbb{F}pp_k^{\Delta}$ (and $R=\mathbb{F}$), so every simple diagonal p-permutation functor S is isomorphic to $S_{G,V}$, where G is a finite group and V is a simple $\mathrm{End}_{\mathbb{F}pp_k^{\Delta}}(G)$-module. In this context, we can assume moreover that G is a group of minimal order such that $S(G)\neq\{0\}$. Then V is actually a simple module for the essential algebra $\mathcal{E}^{\Delta}(G)=\mathrm{End}_{\mathbb{F}pp_k^{\Delta}}(G)/I$ at G, where I is the ideal generated by the morphisms that factor through groups of smaller order.

These considerations motivate the study of the essential algebra $\mathcal{E}^{\Delta}(G)$. We show that this algebra is isomorphic to the essential algebra studied in [7]. As a result, this implies that the essential algebra $\mathcal{E}^{\Delta}(G)$ is non-zero if and only if the group G is of the form $P\rtimes\langle s\rangle$ where P is a p-group and s is a generator of a p'-cyclic group acting faithfully on P. Moreover in that case there is an algebra isomorphism $\mathcal{E}^{\Delta}(G)\cong\big(\mathbb{F}[X]/\Phi_n[X]\big)\rtimes\mathrm{Out}(G)$ where n is the order of s. See Theorem 3.3.

We also study the functor $\mathbb{F}T^{\Delta}$ that sends a finite group G to the Grothendieck group $\mathbb{F}T(G)$ of p-permutation kG-modules. We obtain a description of the lattice of its subfunctors (Theorem 5.11), and deduce that $\mathbb{F}T^{\Delta}$ is semisimple, equal to the direct sum of its simple subfunctors (Theorem 5.14). We describe precisely these simple subfunctors, and show in particular that they are mutually non isomorphic. Next we give a formula for the $\mathbb{F}$-dimension of the evaluations of these simple functors at a finite group G (Theorem 5.16).

Proposition 5.9 and Theorem 5.14 also give a (very partial) answer to the question of knowing if, for a given simple diagonal p-permutation functor S, the groups G of minimal order such that $S(G) \neq \{0\}$ form a single isomorphism class of finite groups. We refer to [11], [12], [13], [3], where similar categories of functors are considered, showing that the answer to the above question is generally negative.

Finally, we prove that the simple functor $S_{1,1}$ that corresponds to the pair $(1,1)$ is isomorphic to the functor that sends a finite group G to the $\mathbb{F}$-linear extension $\mathbb{F}K_0(kG)$ of the Grothendieck group of projective kG-modules (Theorem 5.18).

2. Preliminaries

Let G and H be finite groups. We denote by $p_1 : G\times H \to G$ and $p_2 : G\times H \to H$ the canonical projections. Let $X \leqslant G \times H$ be a subgroup. We define the subgroups $k_1(X) := p_1(X \cap \ker(p_2))$ and $k_2(X) := p_2(X \cap \ker(p_1))$ of $p_1(X)$ and $p_2(X)$, respectively. Note that $k_1(X) \times k_2(X)$ is a normal subgroup of X. Moreover, $k_i(X)$ is a normal subgroup of $p_i(X)$ and one has a canonical isomorphism $X/(k_1(X) \times k_2(X)) \to p_i(X)/k_i(X)$ induced by the projection map p_i for $i = 1, 2$.

Let $\phi : P \to Q$ be an isomorphism between subgroups $P \leqslant G$ and $Q \leqslant H$. Then $\{(\phi(x), x) : x \in P\}$ is a subgroup of $H \times G$ and a subgroup of that form is called a *twisted diagonal* subgroup of $H \times G$. Note that a subgroup $X \leqslant H \times G$ is a twisted diagonal subgroup if and only if $k_1(X) = 1$ and $k_2(X) = 1$.

Let P be a subgroup of G and M be a kG-module. We denote by M^P the k-vector space of P-fixed points of M. If $Q \leqslant P$ is a subgroup, then the map $\mathrm{Tr}_Q^P : M^Q \to M^P$ defined by $\mathrm{Tr}(m) = \sum_{x\in[P/Q]} x \cdot m$ is called the *relative trace map*. The quotient

$$M[P] := M^P / \sum_{Q<P} \mathrm{Tr}_Q^P(M^Q)$$

is called the *Brauer quotient* of M at P. Note that $M[P]$ is a $k\overline{N}_G(P)$-module, where $\overline{N}_G(P) := N_G(P)/P$. We have $M[P] = 0$ if P is not a p-group.

A (kG, kH)-bimodule M can be viewed as a $k(G \times H)$-module via $(g, h) \cdot m := gmh^{-1}$, for $(g, h) \in G \times H$ and $m \in M$. Similarly a $k(G \times H)$-module can be viewed as a (kG, kH)-bimodule. We will usually switch between these two points of views.

Definition 2.1. *Let G be a finite group. A kG-module M is called a permutation module, if M has a G-stable k-basis. A p-permutation kG-module is a kG-module M such that $\mathrm{Res}^G_S M$ is a permutation kS-module for a Sylow p-subgroup S of G.*

For a finite group G we denote by $T(G)$ the Grothendieck group of p-permutation kG-modules with respect to direct sum decompositions. If M is a p-permutation kG-module, then the class of M in $T(G)$ will be abusively denoted by M. The group $T(G)$ has a commutative ring structure induced by the tensor product of modules over k, and $T(G)$ will be called the *ring of p-permutation modules* of G, for short. If H is another finite group, we set $T(G,H) := T(G \times H)$. We denote by $T^{\Delta}(G,H)$ the subgroup of $T(G,H)$ spanned by p-permutation $k(G \times H)$-modules whose indecomposable direct summands have twisted diagonal vertices.

Let $\mathcal{P}_{G,p}$ denote the set of pairs (P,E) where P is a p-subgroup of G and E is a projective indecomposable $k\overline{N}_G(P)$-module. The group G acts on the set $\mathcal{P}_{G,p}$ via conjugation and we denote by $[\mathcal{P}_{G,p}]$ a set of representatives of G-orbits of $\mathcal{P}_{G,p}$. For $(P,E) \in \mathcal{P}_{G,p}$, let $M_{P,E}$ denote the unique (up to isomorphism) indecomposable p-permutation kG-module with the property that $M_{P,E}[P] \cong E$. Note that $M_{P,E}$ has the group P as a vertex [6, Theorem 3.2]. We denote by $\mathcal{P}^{\Delta}_{G\times H,p}$ the set of pairs $(P,E) \in \mathcal{P}_{G\times H,p}$ where P is a twisted diagonal p-subgroup of $G \times H$.

Remark 2.2. *The isomorphism classes of the modules $M_{P,E}$ where $(P,E) \in \mathcal{P}^{\Delta}_{G\times H,p}$ form a $\mathbb{Z}$-basis for $T^{\Delta}(G,H)$.*

Definition 2.3. *[7, Definition 2.3.1] Let (P,s) be a pair where P is a p-group and s is a generator of a p'-cyclic group acting on P. We denote the semidirect product $P \rtimes \langle s\rangle$ by $\langle Ps\rangle$. Let (Q,t) be another such pair. We say that the pairs (P,s) and (Q,t) are isomorphic if there are group isomorphisms $\phi : P \to Q$ and $\psi : \langle s\rangle \to \langle t\rangle$ such that $\psi(s) = q \cdot t$ for some $q \in Q$ and $\phi(s \cdot u) = \psi(s) \cdot \phi(u)$ for all $u \in P$. In that case we write $(P,s) \simeq (Q,t)$.*

Lemma 2.4. *[7, Proposition 2.3.3] Let (P,s) and (Q,t) be two pairs. Then $(P,s) \simeq (Q,t)$ if and only if there is a group isomorphism $f : \langle Ps\rangle \to \langle Qt\rangle$ such that $f(s)$ is conjugate to t.*

Let $\mathcal{Q}_{G,p}$ denote the set of pairs (P,s) where P is a p-subgroup of G and $s \in N_G(P)$ is a p'-element. In that case $\langle Ps\rangle$ denotes the semidirect product $P \rtimes \langle s\rangle$ where the action of $\langle s\rangle$ on P is induced by conjugation. The group G acts on the set $\mathcal{Q}_{G,p}$ and we denote by $[\mathcal{Q}_{G,p}]$ a set of representatives of G-orbits. We denote by $\mathcal{Q}^{\Delta}_{G\times H,p}$ the set of pairs $(P,s) \in \mathcal{Q}_{G\times H,p}$ where P is a twisted diagonal p-subgroup of $G \times H$.

As $\mathbb{F}$ is algebraically closed, we can chose a group isomorphism between the roots of unity in k and the p'-roots of unity in $\mathbb{F}$, and this allows for a definition of ($\mathbb{F}$-valued) Brauer characters. Now for any pair $(P,s) \in \mathcal{Q}_{G,p}$ let $\tau^G_{P,s}$ denote the additive map $T(G) \to \mathbb{F}$ that sends a p-permutation kG-module M to the value of the Brauer character of $M[P]$ at s. The map $\tau^G_{P,s}$ is a ring homomorphism and it extends to an $\mathbb{F}$-algebra homomorphism $\tau^G_{P,s} : \mathbb{F} \otimes_{\mathbb{Z}} T(G) \to \mathbb{F}$. The set $\{\tau^G_{P,s} : (P,s) \in [\mathcal{Q}_{G,p}]\}$ is the set of all species from $\mathbb{F}T(G) := \mathbb{F} \otimes_{\mathbb{Z}} T(G)$ to $\mathbb{F}$ [5, Proposition 2.18].

The commutative algebra $\mathbb{F}T(G)$ is split semisimple and its primitive idempotents $F^G_{P,s}$ are indexed by pairs $(P,s) \in [\mathcal{Q}_{G,p}]$ [5, Corollary 2.19]. If $\phi : \langle s \rangle \to k^\times$ is a group homomorphism, we denote by k_ϕ the $k\langle s \rangle$-module k on which the element s acts as multiplication by $\phi(s)$. Let $\widehat{\langle s \rangle} = \mathrm{Hom}(\langle s \rangle, k^\times)$ denote the set of group homomorphisms. By [5, Theorem 4.12] we have the idempotent formula

$$F^G_{P,s} = \frac{1}{|P||s||C_{\overline{N}_G(P)}(s)|} \sum_{\substack{\varphi \in \widehat{\langle s \rangle} \\ L \leqslant \langle Ps \rangle \\ PL = \langle Ps \rangle}} \tilde{\varphi}(s^{-1})|L|\mu(L, \langle Ps \rangle)\mathrm{Ind}^G_L k^{\langle Ps \rangle}_{L,\varphi},$$

where $k^{\langle Ps \rangle}_{L,\varphi} = \mathrm{Res}^{\langle Ps \rangle}_L \mathrm{Inf}^{\langle Ps \rangle}_{\langle s \rangle} k_\varphi$, and $\tilde{\varphi}$ is the Brauer character of k_φ.

By [7, Proposition 2.7.8] we have another formula

$$F^G_{P,s} = \frac{1}{|C_{N_G(P)}(s)|} \sum_{\substack{\varphi \in \widehat{\langle s \rangle} \\ L \leqslant P \\ L^s = L}} \tilde{\varphi}(s^{-1})|C_L(s)|\mu\big((L,P)^s\big)\mathrm{Ind}^G_{\langle Ls \rangle} k^{\langle Ps \rangle}_{\langle Ls \rangle,\varphi}.$$

Here $\mu\big((-,-)^s\big)$ is the Möbius function of the poset of s-stable subgroups of P.

Lemma 2.5. *For finite groups G and H, the set $\{F^{G\times H}_{P,s} : (P,s) \in [\mathcal{Q}^\Delta_{G\times H,p}]\}$ of primitive idempotents form an $\mathbb{F}$-basis for the split semisimple algebra $\mathbb{F}T^\Delta(G,H)$.*

Proof. First we will show that we have $F^{G\times H}_{P,s} \in \mathbb{F}T^\Delta(G,H)$ whenever $(P,s) \in [\mathcal{Q}^\Delta_{G\times H,p}]$. Let $\varphi \in \widehat{\langle s \rangle}$ and $L \leqslant \langle Ps \rangle$. It suffices to show that $\mathrm{Ind}^G_L k^{\langle Ps \rangle}_{L,\varphi} \in \mathbb{F}T^\Delta(G,H)$. Since P acts trivially on $\mathrm{Inf}^{\langle Ps \rangle}_{\langle s \rangle} k_\varphi$, the subgroup P is contained in a vertex of k_φ considered as a $k\langle Ps \rangle$-module. But since P is the Sylow p-subgroup of $\langle Ps \rangle$, it follows that P is the vertex of k_φ. Therefore the module $k^{\langle Ps \rangle}_{L,\varphi} = \mathrm{Res}^{\langle Ps \rangle}_L \mathrm{Inf}^{\langle Ps \rangle}_{\langle s \rangle} k_\varphi$ has a vertex contained in $L \cap {}^xP \leqslant P$ for some $x \in \langle Ps \rangle$. Since a subgroup of twisted diagonal subgroup is again twisted diagonal, this means that $k^{\langle Ps \rangle}_{L,\varphi}$ has twisted diagonal vertices. This shows that $\mathrm{Ind}^G_L k^{\langle Ps \rangle}_{L,\varphi} \in \mathbb{F}T^\Delta(G,H)$ as desired. Now since the

$\mathbb{F}$-dimension of $\mathbb{F}T^\Delta(G,H)$ is equal to the cardinality of $[\mathcal{P}^\Delta_{G\times H,p}]$, which is equal to the cardinality of $[\mathcal{Q}^\Delta_{G\times H,p}]$, it follows that the set $\{F^{G\times H}_{P,s} : (P,s)\in[\mathcal{Q}^\Delta_{G\times H,p}]\}$ of primitive idempotents form an $\mathbb{F}$-basis for $\mathbb{F}T^\Delta(G,H)$. □

Let G, H and L be finite groups. If X is a (kG,kH)-bimodule and Y is a (kH,kL)-bimodule, then $X\circ Y := X\otimes_{kH} Y$ is a (kG,kL)-bimodule. Extending this product by $\mathbb{F}$-bilinearity, we get a map

$$\mathbb{F}T(G,H)\circ\mathbb{F}T(H,L)\to\mathbb{F}T(G,L).$$

Note that this induces a map

$$\mathbb{F}T^\Delta(G,H)\circ\mathbb{F}T^\Delta(H,L)\to\mathbb{F}T^\Delta(G,L)$$

which is used to define the composition of morphisms in the following category.

Definition 2.6. *Let $\mathbb{F}pp^\Delta_k$ be the category with*

- *objects: finite groups*
- $\mathrm{Mor}_{\mathbb{F}pp^\Delta_k}(G,H)=\mathbb{F}\otimes_{\mathbb{Z}} T^\Delta(H,G)=\mathbb{F}T^\Delta(H,G)$.

An $\mathbb{F}$-linear functor from $\mathbb{F}pp^\Delta_k$ to $\mathbb{F}$-Mod is called a *diagonal p-permutation functor*. Diagonal p-permutation functors form an abelian category $\mathcal{F}^\Delta_{pp_k}$.

3. The Essential Algebra

For a finite group G, the quotient algebra

$$\mathcal{E}^\Delta(G) := \mathbb{F}T^\Delta(G,G)/\Big(\sum_{|H|<|G|}\mathbb{F}T^\Delta(G,H)\circ\mathbb{F}T^\Delta(H,G)\Big)$$

is called the *essential algebra* of G.

By [7, Proposition 4.1.2 and Theorem 4.1.12] the algebra

$$\mathcal{E}(G) := \mathbb{F}T(G,G)/\Big(\sum_{|H|<|G|}\mathbb{F}T(G,H)\circ\mathbb{F}T(H,G)\Big)$$

is non-zero if and only if there exists a pair (P,s) in G such that $G=\langle Ps\rangle$ and $C_{\langle s\rangle}(P)=1$. In that case, we also have, in the notation of [7, Theorem 4.1.12], an algebra isomorphism

$$\mathcal{E}(G)\cong\big(\mathbb{F}[X]/\Phi_n[X]\big)\rtimes\mathrm{Out}(G)$$

where n is the order of s, but we will not use this isomorphism in this paper.

Note that the inclusion map $\mathbb{F}T^\Delta(G,G) \hookrightarrow \mathbb{F}T(G,G)$ induces a map

$$\Theta : \mathcal{E}^\Delta(G) \to \mathcal{E}(G).$$

We will show that this map is an algebra isomorphism.

Let $\varphi \in \mathrm{Aut}(G)$ be an automorphism and $\lambda : G/O_p(G) \to k^\times$ be a character, where $O_p(G)$ denotes the largest normal p-subgroup of G. We define a (kG,kG)-bimodule structure on kG, denoted by $kG_{\varphi,\lambda}$, via

$$a \cdot g \cdot b := \lambda(b) a g \varphi(b)$$

for $a, b, g \in G$.

Let $\langle Rt \rangle$ be a twisted diagonal subgroup of $G \times G$ with $p_1(\langle Rt \rangle) = G$ and $p_2(\langle Rt \rangle) = G$. Let also $\eta : p_1(\langle Rt \rangle) \to p_2(\langle Rt \rangle)$ be the canonical isomorphism. Then by [7, Section 4.1.2] we have an isomorphism

$$\mathrm{Ind}_{\langle Rt \rangle}^{G\times G} k_{\langle Rt \rangle,\varphi}^{\langle Rt \rangle} \cong kG_{\eta^{-1},\varphi^{-1}}$$

of (kG,kG)-bimodules. Again by [7, Section 4.1.2] the algebra $\mathcal{E}(G)$ is generated by the images of $kG_{\varphi,\lambda}$.

Proposition 3.1. *If the essential algebra $\mathcal{E}^\Delta(G)$ of a finite group G is non-zero, then there exists a pair (P,s) in G such that $G = \langle Ps \rangle$ and $C_{\langle s \rangle}(P) = 1$.*

Proof. Let (Q,t) be a pair contained in $G \times G$ such that Q is a twisted diagonal subgroup and recall the idempotent formula

$$F_{Q,t}^{G\times G} = \frac{1}{|C_{N_{G\times G(Q)}(t)}|} \sum_{\substack{\varphi \in \widehat{\langle t \rangle} \\ L \leqslant Q \\ L^t = L}} \tilde{\varphi}(t^{-1}) |C_L(t)| \mu((L,Q)^t) \mathrm{Ind}_{\langle Lt \rangle}^{G\times G} k_{L,\varphi}^{\langle Qt \rangle}.$$

By [7, Lemma 2.5.9] we have an isomorphism

$$\begin{aligned}\mathrm{Ind}_{\langle Lt \rangle}^{G\times G} k_{L,\varphi}^{\langle Qt \rangle} &\cong \mathrm{Ind}_{p_1(\langle Lt \rangle)}^{G} \otimes_{p_1(\langle Lt \rangle)} \mathrm{Ind}_{\langle Lt \rangle}^{p_1(\langle Lt \rangle)\times p_2(\langle Lt \rangle)} (k_{L,\varphi}^{\langle Qt \rangle}) \otimes_{p_2(\langle Lt \rangle)} \mathrm{Res}_{p_2(\langle Lt \rangle)}^{G} \\ &\cong kG \otimes_{p_1(\langle Lt \rangle)} \mathrm{Ind}_{\langle Lt \rangle}^{p_1(\langle Lt \rangle)\times p_2(\langle Lt \rangle)} (k_{L,\varphi}^{\langle Qt \rangle}) \otimes_{p_2(\langle Lt \rangle)} kG\end{aligned}$$

of (kG,kG)-bimodules. As (kG,kG)-bimodule, we have the isomorphism $kG \cong \mathrm{Ind}_{\Delta G}^{G\times G} k$. Thus as $(kG, kp_1(\langle Lt \rangle))$-bimodule we have,

$$\mathrm{Res}_{G\times p_1(\langle Lt \rangle)}^{G\times G} kG \cong \mathrm{Res}_{G\times p_1(\langle Lt \rangle)}^{G\times G} \mathrm{Ind}_{\Delta G}^{G\times G} k \cong \mathrm{Ind}_{\Delta(p_1(\langle Lt \rangle))}^{G\times p_1(\langle Lt \rangle)} \mathrm{Res}_{\Delta(p_1(\langle Lt \rangle))}^{\Delta(G)} k.$$

Therefore as $k\big(G \times p_1(\langle Lt\rangle)\big)$-module, the indecomposable direct summands of kG have vertices contained in $\Delta(p_1(\langle Lt\rangle))$. Similary, one can show that the indecomposable direct summands of kG as $k\big(p_2(\langle Lt\rangle) \times G\big)$-module, have vertices contained in $\Delta(p_2(\langle Lt\rangle))$. We also know that the module $k_{L,\varphi}^{\langle Qt\rangle}$, and hence the indecomposable direct summands of $\mathrm{Ind}_{\langle Lt\rangle}^{p_1(\langle Lt\rangle)\times p_2(\langle Lt\rangle)}(k_{L,\varphi}^{\langle Qt\rangle})$, have twisted diagonal vertices. Now suppose $\mathcal{E}^\Delta(G)$ is non-zero. Then there is an idempotent $F_{Q,t}^{G\times G}$ whose image in $\mathcal{E}^\Delta(G)$ is non-zero. Therefore the argument above shows that there is a pair (Q,t) in $G\times G$ such that $p_1(\langle Qt\rangle) = G$ and $p_2(\langle Qt\rangle) = G$. This implies that there is a p-subgroup P of G and a p'-element s of G that normalises P such that $G = \langle Ps\rangle$. Now we will show that in that case we have $C_{\langle s\rangle}(P) = 1$.
Let $\overline{G} := G/C_{\langle s\rangle}(P)$, $Q := \{(u,\overline{u}) : u \in P\} \leqslant G \times \overline{G}$ and $Q' := \{(\overline{u},u) : u \in P\} \leqslant \overline{G}\times G$. Then by [7, Proof of Proposition 4.1.2] we have an isomorphism of (kG,kG)-bimodules between kG and a direct sum

$$\bigoplus_i \mathrm{Indinf}_{\overline{N}_{G\times\overline{G}}(Q)}^{G\times\overline{G}} F_i \otimes_{k\overline{G}} \mathrm{Indinf}_{\overline{N}_{\overline{G}\times G}(Q')}^{\overline{G}\times G} F_i'$$

where $\mathrm{Indinf}_{\overline{N}_{G\times\overline{G}}(Q)}^{G\times\overline{G}} = \mathrm{Ind}_{N_{G\times\overline{G}}(Q)}^{G\times\overline{G}} \circ \mathrm{Inf}_{\overline{N}_{G\times\overline{G}}(Q)}^{N_{G\times\overline{G}}(Q)}$, and F_i and F_i' are projective indecomposable $k\overline{N}_{G\times\overline{G}}(Q)$-modules and $k\overline{N}_{\overline{G}\times G}(Q')$-modules respectively. Now since F_i is projective indecomposable, it has the trivial group as vertex. So $\mathrm{Inf}_{\overline{N}_{G\times\overline{G}}(Q)}^{N_{G\times\overline{G}}(Q)} F_i$ has the group Q as a vertex. Note that the group Q is twisted diagonal. Therefore indecomposable direct summands of $\mathrm{Indinf}_{\overline{N}_{G\times\overline{G}}(Q)}^{G\times\overline{G}} F_i$ have twisted diagonal vertices, i.e. $\mathrm{Indinf}_{\overline{N}_{G\times\overline{G}}(Q)}^{G\times\overline{G}} F_i \in \mathbb{F}T^\Delta(G,\overline{G})$. Similarly, we have $\mathrm{Indinf}_{\overline{N}_{\overline{G}\times G}(Q')}^{\overline{G}\times G} F_i' \in \mathbb{F}T^\Delta(\overline{G},G)$. Now since $\mathcal{E}^\Delta(G) \neq 0$, the image of identity element $kG \in \mathbb{F}T^\Delta(G,G)$ in $\mathcal{E}^\Delta(G)$ is non-zero. Hence we have $\overline{G} = G$, i.e. $C_{\langle s\rangle}(P) = 1$. □

Suppose we have $G = \langle Ps\rangle$ and $C_{\langle s\rangle}(P) = 1$. The essential algebra $\mathcal{E}^\Delta(G)$ is generated by the images of the primitive idempotents

$$F_{Q,t}^{G\times G} = \frac{1}{\mid C_{N_{G\times G}(Q)}(t)\mid} \sum_{\substack{\varphi\in\widehat{\langle t\rangle}\\ L\leqslant Q\\ L^t=L}} \tilde{\varphi}(t^{-1})|C_L(t)|\mu((L,Q)^t)\mathrm{Ind}_{\langle Lt\rangle}^{G\times G} k_{L,\varphi}^{\langle Qt\rangle}$$

where Q is a twisted diagonal subgroup of $G\times G$. By [7, Lemma 2.5.9], if the image of $\mathrm{Ind}_{\langle Lt\rangle}^{G\times G} k_{L,\varphi}^{\langle Qt\rangle}$ is non-zero, then we must have that $p_1(\langle Lt\rangle) = G = p_2(\langle Lt\rangle)$. Write $t = (u,v)$. Then $p_1(\langle Lt\rangle) = \langle p_1(L)u\rangle$ and $p_2(\langle Lt\rangle) = \langle p_2(L)v\rangle$. Therefore we have

$|u| = |v| = |s|$. Being a subgroup of twisted diagonal subgroup Q, the group L itself is also twisted diagonal. Since $k_1(L) = k_2(L) = 1$ and $|u| = |v| = |s|$, we have $k_1(\langle Lt\rangle) = k_2(\langle Lt\rangle) = 1$. This shows that the subgroup $\langle Lt\rangle$ is twisted diagonal and $p_1(\langle Lt\rangle) = G = p_2(\langle Lt\rangle)$. Since the images of $\mathrm{Ind}_{\langle Lt\rangle}^{G\times G} k_{L,\varphi}^{\langle Qt\rangle}$ in $\mathcal{E}(G)$ with $\langle Lt\rangle$ satisfying these properties, generate the non-zero algebra $\mathcal{E}(G)$, this shows that the algebra $\mathcal{E}^{\Delta}(G)$ is also non-zero and the map $\Theta : \mathcal{E}^{\Delta}(G) \to \mathcal{E}(G)$ is surjective. Thus we have proved the following:

Proposition 3.2. *The essential algebra $\mathcal{E}^{\Delta}(G)$ is non-zero if and only if there is a pair (P, s) in G such that $G = \langle Ps\rangle$ and $C_{\langle s\rangle}(P) = 1$. Moreover the map $\Theta : \mathcal{E}^{\Delta}(G) \to \mathcal{E}(G)$ is surjective.*

Suppose we have $G = \langle Ps\rangle$ for some pair and $C_{\langle s\rangle}(P) = 1$. We will show that the map $\Theta : \mathcal{E}^{\Delta}(G) \to \mathcal{E}(G)$ is also injective.
Suppose an element $\sum \overline{r_{\varphi,\alpha} kG_{\varphi,\alpha}} \in \mathcal{E}^{\Delta}(G)$ is mapped to zero by Θ. We must show that the element $\sum \overline{r_{\varphi,\alpha} kG_{\varphi,\alpha}}$ of $\mathcal{E}(G)$ is zero. Write

$$\sum r_{\varphi,\alpha} kG_{\varphi,\alpha} = \sum_{|H|<|G|} t_{H,U_H,V_H} U_H \otimes_{kH} V_H$$

for some (kG, kH)-bimodule U_H and (kH, kG)-bimodule V_H and some constants $t_{H,U_H,V_H} \in \mathbb{F}$. Suppose the coefficient t_{H,U_H,V_H} is non-zero for some group H. Then as in [7] we can assume that $H = \langle Rt\rangle$ for some pair (R, t) and that the modules U_H and V_H are indecomposable. By [7, Section 4.1] one has

$$U_H \otimes_{kH} V_H \cong \mathrm{Indinf}_{\overline{N}_{G\times G}(\Delta(P))}^{G\times G} \bigoplus_i \big(kZ(P) \otimes k_{\lambda_i}\big)^{n_i}$$

where λ_i is a character of $\langle s\rangle$ and $n_i \in \mathbb{N}$. Again by [7, Section 4.1] each summand $kZ(P) \otimes k_{\lambda_i}$ is a projective indecomposable $k\overline{N}_{G\times G}(\Delta(P))$-module. This shows that if the the coefficient t_{H,U_H,V_H} is non-zero, then the indecomposable direct summands of the bimodule $U_H \otimes_{kH} V_H$ have twisted diagonal vertices. Therefore the element $\sum \overline{r_{\varphi,\alpha} kG_{\varphi,\alpha}}$ is zero in $\mathcal{E}^{\Delta}(G)$. This proves that the map $\Theta : \mathcal{E}^{\Delta}(G) \to \mathcal{E}(G)$ is injective. We summarise our results as a theorem below.

Theorem 3.3. *The essential algebra $\mathcal{E}^{\Delta}(G)$ is non-zero if and only if there is a pair (P, s) in G such that $G = \langle Ps\rangle$ and $C_{\langle s\rangle}(P) = 1$. In that case, the algebra $\mathcal{E}^{\Delta}(G)$ is isomorphic to the algebra $\big(\mathbb{F}[X]/\Phi_n[X]\big) \rtimes \mathrm{Out}(G)$ where n is the order of s.*

4. D^{Δ}-pairs

Let $H \leqslant G$ be a subgroup. The (kG, kH)-bimodule kG is denoted by Ind_H^G and (kH, kG)-bimodule kG is denoted by Res_H^G. Similarly, if $N \trianglelefteq G$ is a normal subgroup, the $(kG/N, kG)$-bimodule kG/N is denoted by $\mathrm{Def}_{G/N}^G$ and $(kG, kG/N)$-bimodule kG/N is denoted by $\mathrm{Inf}_{G/N}^G$. This notation is consistent with our previous use of induction, restriction, inflation and deflation symbols, in the sense that for example, if M is a kH-module, then the induced module $\mathrm{Ind}_H^G M$ is isomorphic to $\mathrm{Ind}_H^G \otimes_{kH} M$.

We have the following lemma due to [5] and [7].

Lemma 4.1. *(i) Let $(P, s) \in \mathcal{Q}_{G,p}$ be a pair and $H \leqslant G$ be a subgroup. Then we have*

$$\mathrm{Res}_H^G F_{P,s}^G = \sum_{Q,t} F_{Q,t}^H$$

where (Q, t) runs over a set of representatives of H-conjugacy classes of G-conjugates of (P, s) contained in H.

(ii) Let $(Q, t) \in \mathcal{Q}_{H,p}$ be a pair and $H \leqslant G$ be a subgroup. Then we have

$$\mathrm{Ind}_H^G F_{Q,t}^H = |N_G(Q, t) : N_H(Q, t)| F_{Q,t}^G.$$

(iii) Let $N \trianglelefteq G$ and $(P, s) \in \mathcal{Q}_{G/N,p}$. Then

$$\mathrm{Inf}_{G/N}^G F_{P,s}^{G/N} = \sum_{Q,t} F_{Q,t}^G$$

where (Q, t) runs over a set of representatives of G-conjugacy classes of pairs in $\mathcal{Q}_{G,p}$ such that $QN/N = {}^{\overline{g}}P$ and $\overline{t} = {}^{g}s$ for some $\overline{g} \in G/N$.

(iv) Let $N \trianglelefteq G$ and $(P, s) \in \mathcal{Q}_{G,p}$. Then

$$\mathrm{Def}_{G/N}^G F_{P,s}^G = m_{P,s,N} \cdot F_{Q,t}^{G/N}$$

for some pair $(Q, t) \in \mathcal{Q}_{G/N,p}$ and a constant $m_{P,s,N} \in \mathbb{F}$.
If $G = \langle Ps \rangle$ then

$$\mathrm{Def}_{G/N}^G F_{P,s}^G = m_{P,s,N} \cdot F_{PN/N,\overline{s}}^{G/N}.$$

Proof. See [5, Proposition 3.1. and Proposition 3.2.] for (i) and (ii), [7, Proposition 3.1.3] for (iii) and [7, Lemma 3.1.4 and Proposition 3.1.5] for (iv). □

Lemma 4.2. *Let $N \trianglelefteq G$ be a normal subgroup of G.*

(i) We have $\mathrm{Def}^G_{G/N} \in \mathbb{F}T^{\Delta}(G/N, G)$ if and only if N is a p'-group.

(ii) We have $\mathrm{Inf}^G_{G/N} \in \mathbb{F}T^{\Delta}(G, G/N)$ if and only if N is a p'-group.

Proof. (i) Let $Q \leqslant (G/N)\times G$ be a maximal vertex of an indecomposable direct summand of the $(kG/N, kG)$-bimodule kG/N. Equivalently Q is a maximal p-subgroup having a fixed point on the set G/N. Suppose $(aN, b) \in Q$ stabilises a basis element gN of kG/N. Then we have $(aN)gNb^{-1} = gN$ which implies that $a^g \cdot b^{-1} \in N$. Since the vertices of an indecomposable module are conjugate, we may assume that $g = 1$. Thus, up to conjugacy, Q is a Sylow p-subgroup of

$$H = \{(aN, b) : ab^{-1} \in N\} \leqslant (G/N) \times G.$$

Note that $k_1(Q) = k_1(H) = 1$ and $k_2(Q)$ is a Sylow p-subgroup of N. Hence Q is twisted diagonal if and only if N is a p'-group. The result follows.
(ii) Similar. □

Let (P, s) be a pair and suppose $G = \langle Ps\rangle$. Then by [7, Corollary 3.1.9] for any normal subgroup N of G, we have the following formula for the constant $m_{P,s,N}$:

$$m_{P,s,N} = \frac{|s|}{|N \cap \langle s\rangle||C_G(s)|} \sum_{\substack{Q\leqslant P\\ Q^s=Q\\ \langle Qs\rangle N=G}} |C_Q(s)|\mu\big((Q,P)^s\big).$$

Lemma 4.3. *Let (P, s) be a pair and suppose $G = \langle Ps\rangle$. Then for any normal p'-subgroup N of G we have*

$$m_{P,s,N} = \frac{1}{|N|}.$$

Proof. First observe that since N is a p'-group, we have $N \leqslant C_{\langle s\rangle}(P)$. For any subgroup Q of P the condition $\langle Qs\rangle N = \langle Ps\rangle$ implies that $|Q| = |P|$ and hence $Q = P$. Therefore the formula above becomes

$$m_{P,s,N} = \frac{|s||C_P(s)|}{|N||C_G(s)|} = \frac{1}{|N|}.$$

□

Definition 4.4. *A pair (P, s) is called D^Δ-pair if* $\mathrm{Def}^{\langle Ps\rangle}_{\langle Ps\rangle/N} F^{\langle Ps\rangle}_{P,s} = 0$ *for any nontrivial normal p'-subgroup N of $\langle Ps\rangle$.*

Lemma 4.5. *Let (P, s) be a pair. Then (P, s) is a D^Δ-pair if and only if the group $\langle Ps\rangle$ does not have any nontrivial normal p'-subgroup, that is, if and only if $C_{\langle s\rangle}(P) = 1$.*

Proof. By Lemma 4.3, for any normal p'-subgroup $N \trianglelefteq \langle Ps\rangle$ we have $m_{P,s,N} = 1/|N|$. Therefore (P, s) is a D^Δ-pair if and only if the group $\langle Ps\rangle$ does not have any nontrivial normal p'-subgroup. The result follows. □

5. The functor $\mathbb{F}T^\Delta$

By [2], the simple diagonal p-permutation functors are parametrized by the pairs (G, V) where G is a finite group and V is a simple $\mathcal{E}^\Delta(G)$-module. Note that this implies $\mathcal{E}^\Delta(G) \neq 0$.

For a simple $\mathcal{E}^\Delta(G)$-module V, we define two functors in $\mathbb{F}pp^\Delta_k$ by:

$$L_{G,V}(H) := \mathbb{F}T^\Delta(H, G) \otimes_{\mathcal{E}^\Delta(G)} V$$

and

$$J_{G,V}(H) := \left\{ \sum_i \phi_i \otimes v_i \in L_{G,V} : \forall \psi \in \mathbb{F}T^\Delta(G, H), \sum_i (\psi \circ \phi_i) \cdot v_i = 0 \right\},$$

for any finite group H. The action of morphisms in $\mathbb{F}pp^\Delta_k$ on these evaluations is given by left composition. The functor $J_{G,V}$ is the unique maximal subfunctor of $L_{G,V}$, so the quotient

$$S_{G,V} := L_{G,V}/J_{G,V}$$

is a simple functor [2].

Let $\mathbb{F}T^\Delta : \mathbb{F}pp^\Delta_k \to \mathbb{F}\text{-Mod}$ be the functor given by

- $\mathbb{F}T^\Delta(G) := \mathbb{F} \otimes_{\mathbb{Z}} T(G) = \mathbb{F}T(G)$,
- $\mathbb{F}T^\Delta(X) : \mathbb{F}T(G) \to \mathbb{F}T(H), M \mapsto X \otimes_{kH} M$ for any $X \in \mathbb{F}T^\Delta(H, G)$.

For any kG-module X, we denote by $\widetilde{X}$ the (kG, kG)-bimodule $k(G \times X)$ where the action of kG-kG is given by

$$a \cdot (g, x) \cdot b^{-1} := (agb, b^{-1}x)$$

for all $a, b, g \in G$ and $x \in X$. We have an isomorphism of (kG, kG)-bimodules

$$\widetilde{X} \cong \mathrm{Ind}_{\delta(G)}^{G\times G^{op}} \mathrm{Iso}(\delta)(X)$$

where $\delta : G \to G \times G^{op}$, $g \mapsto (g, g^{-1})$. See [7, Definition 2.5.17]. Note that the image $\delta(G)$ of G in $G \times G^{op}$ is a twisted diagonal subgroup. If X is an indecomposable p-permutation kG-module with a vertex Q, then any vertex of an indecomposable direct summand of $\widetilde{X}$ is contained in $\delta(Q)$, up to conjugation. Therefore for any $X \in \mathbb{F}T(G)$ we have $\widetilde{X} \in \mathbb{F}T^{\Delta}(G, G)$.

Lemma 5.1. *Let F be a subfunctor of $\mathbb{F}T^{\Delta}$. Then for any finite group G, the $\mathbb{F}$-vector space $F(G)$ is an ideal of the algebra $\mathbb{F}T^{\Delta}(G)$ of p-permutation modules.*

Proof. Let $Y \in F(G)$ and assume X is a p-permutation kG-module. By [7, Proposition 2.5.18] we have an isomorphism $X \otimes_k Y \cong \widetilde{X} \otimes_{kG} Y$ of kG-modules. Since $\widetilde{X} \in \mathbb{F}T^{\Delta}(G, G)$ and F is a functor, we have $\widetilde{X} \otimes_{kG} Y \in F(G)$. This shows that $F(G)$ is an ideal of $\mathbb{F}T^{\Delta}(G)$. □

Definition 5.2. *For any pair (P, s) let $\mathbf{e}_{P,s}$ denote the subfunctor of $\mathbb{F}T^{\Delta}$ generated by the idempotent $F_{P,s}^{\langle Ps\rangle} \in \mathbb{F}T^{\Delta}\big(\langle Ps\rangle\big)$.*

Proposition 5.3. *Let F be a subfunctor of $\mathbb{F}T^{\Delta}$. Then we have*

$$F = \sum_{\mathbf{e}_{P,s} \leqslant F} \mathbf{e}_{P,s}.$$

Proof. Since F is a subfunctor, we have

$$\sum_{\mathbf{e}_{P,s} \leqslant F} \mathbf{e}_{P,s} \leqslant F.$$

Now let G be a finite group, and $u = \sum_{(P,s)} \lambda_{P,s} F_{P,s}^G$, where (P, s) runs in a set of representatives of G-conjugacy classes of $\mathcal{Q}_{G,p}$, and $\lambda_{P,s} \in \mathbb{F}$. Then $F_{P,s}^G \cdot u = \lambda_{P,s} F_{P,s}^G \in F(G)$, since $F(G)$ is an ideal of $\mathbb{F}T^{\Delta}(G)$. Hence $F_{P,s}^G \in F(G)$ if $\lambda_{P,s} \neq 0$. In this case we have $\mathrm{Res}_{\langle Ps\rangle}^G F_{P,s}^G \in F\big(\langle Ps\rangle\big)$, which implies by Lemma 4.1 that $F_{P,s}^{\langle Ps\rangle} \in F\big(\langle Ps\rangle\big)$. This shows that $\mathbf{e}_{P,s} \leqslant F$. By Lemma 4.1 again, $F_{P,s}^G$ is a non zero scalar multiple of $\mathrm{Ind}_{\langle Ps\rangle}^G F_{P,s}^{\langle Ps\rangle}$, so $F_{P,s}^G \in \mathbf{e}_{P,s}(G)$, which gives finally

$$u \in \sum_{\mathbf{e}_{P,s} \leqslant F} \mathbf{e}_{P,s}(G).$$

Therefore we have

$$F = \sum_{\mathbf{e}_{P,s} \leqslant F} \mathbf{e}_{P,s}$$

as desired. $\square$

Proposition 5.4. *Let $(P_i, s_i)_{i\in I}$ be a set of pairs for an indexing set I. Then for any pair (Q,t) we have $\mathbf{e}_{Q,t} \leqslant \sum_{i\in I} \mathbf{e}_{P_i,s_i}$ if and only if $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P_i,s_i}$ for some $i \in I$.*

Proof. If $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P_i,s_i}$ for some $i \in I$, then we obviously have $\mathbf{e}_{Q,t} \leqslant \sum_{i\in I} \mathbf{e}_{P_i,s_i}$. Conversely assume we have $\mathbf{e}_{Q,t} \leqslant \sum_{i\in I} \mathbf{e}_{P_i,s_i}$. Then $\mathbf{e}_{Q,t}\big(\langle Qt\rangle\big) \leqslant \sum_{i\in I} \mathbf{e}_{P_i,s_i}\big(\langle Qt\rangle\big)$ and so $F_{Q,t}^{\langle Qt\rangle} \in \sum_{i\in I} \mathbf{e}_{P_i,s_i}\big(\langle Qt\rangle\big)$. Since $F_{Q,t}^{\langle Qt\rangle}$ is a primitive idempotent and since $\mathbf{e}_{P_i,s_i}\big(\langle Qt\rangle\big)$ is an ideal of $\mathbb{F}T^{\Delta}\big(\langle Qt\rangle\big)$ it follows that we have $F_{Q,t}^{\langle Qt\rangle} \in \mathbf{e}_{P_i,s_i}\big(\langle Qt\rangle\big)$ for some $i \in I$ and hence $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P_i,s_i}$. $\square$

Let G be a finite group and $(P,s) \in \mathcal{Q}_{G,p}$ be a pair such that $G = \langle Ps\rangle$. Let also $(Q,t) \in \mathcal{Q}^{\Delta}_{H\times G,p}$ for a finite group H. Suppose that $\eta : p_1(Q) \to p_2(Q)$ is the canonical isomorphism. Up to conjugation in $H \times G$, we can assume $t = (u, s^j)$. By [7, Section 3.2] if $p_2(\langle Qt\rangle) \neq G$, then the product $F_{Q,t}^{H\times G} \otimes_{kG} F_{P,s}^G$ is zero. So assume that we have $p_2(\langle Qt\rangle) = G$. This implies that we have $p_2(Q) = P$ and $|s^j| = |s|$. Then since $k_1(Q) = k_2(Q) = 1$, this implies that we have $p_1(Q) \cong P$. Since the group Q is t-stable, the isomorphism $\eta : p_1(Q) \to P$ commutes with conjugations by u and s^j. Now [7, Equation (3.3), Section 3.2] implies that as kH-module the product $F_{Q,t}^{H\times G} \otimes_{kG} F_{P,s}^G$ is equal to

$$\frac{1}{|C_{N_{H\times G}}(Q)(t)||C_G(s)|} \sum_{\substack{\varphi\in\overline{\langle t\rangle} \\ \psi\in\overline{\langle s\rangle} \\ \varphi^{|u|}\psi^{j|u|}=1}} \tilde{\varphi}(t)^{-1}\tilde{\psi}(s)^{-1}|C_Q(t)| \sum_{\substack{J\leqslant p_1(Q) \\ J^u=J}} \sigma(J)\mathrm{Ind}_{\langle Ju\rangle}^H(k_{\langle Ju\rangle,\phi}^{\langle p_1(Q)u\rangle})$$

where $\sigma(J) := \sum_{\substack{L\leqslant P \\ L^s=L \\ \eta(J)=L}} |C_L(s)|\mu\big((L,P)^s\big)$ and $\phi(u) := \varphi(u, s^j)\psi(s)^j$.

Suppose we have $H = \langle P's'\rangle$ for a pair (P', s'). Then by [7, Lemma 2.7.6] if $\tau_{P',s'}^H(F_{Q,t}^{H\times G} \otimes_{kG} F_{P,s}^G) \neq 0$, then we must have $p_1(Q) = P'$ and $|u| = |s'|$. This implies in particular that we must have $P' \cong P$. Moreover again by [7, Lemma 2.7.6] we have $\tau_{P',s'}^H\big(\mathrm{Ind}_{\langle Ju\rangle}^H(k_{\langle Ju\rangle,\phi}^{\langle p_1(Q)u\rangle})\big) = 0$ if $J \neq P'$. Therefore if we have $P' \cong P$ then $\tau_{P',s'}^H(F_{Q,t}^{H\times G} \otimes_{kG} F_{P,s}^G)$ is equal to

$$\frac{1}{|C_{N_{H\times G}}(Q)(t)||C_G(s)|} \sum_{\substack{\varphi\in\overline{\langle t\rangle} \\ \psi\in\overline{\langle s\rangle} \\ \varphi^{|u|}\psi^{j|u|}=1}} \tilde{\varphi}(t)^{-1}\tilde{\psi}(s)^{-1}|C_Q(t)||C_P(s)|\tilde{\phi}(s').$$

This shows that if we have $\mathbb{F}T^{\Delta}\big(\langle P's'\rangle, \langle Ps\rangle\big) \otimes_{k\langle Ps\rangle} F_{P,s}^{\langle Ps\rangle} \neq 0$, then there is an isomorphism $\eta : P' \to P$ and a p'-element $(u, s^j) \in \langle P's'\rangle \times \langle Ps\rangle$ such that $\eta \circ c_u = c_{s^j} \circ \eta$ and $|u| = |s'|$, $|s^j| = |s|$. In that case, assume further that $C_{\langle s\rangle}(P) = 1$. Then we have $|c_s| = |s|$ and $|c_{s^j}| = |s^j|$. Since we have $\eta \circ c_u = c_{s^j} \circ \eta$ it follows that $|c_u| = |c_{s^j}|$. Therefore we have $|s| \mid |s'|$. But then [7, Proposition 2.3.6] implies that there is a surjective group homomorphism $\overline{\eta} : \langle P's'\rangle \to \langle Ps\rangle$ that induces an isomorphism of pairs $\big(P' \ker(\overline{\eta})/\ker(\overline{\eta}), s' \ker(\overline{\eta})\big) \simeq (P, s)$. Note that since $|P'| = |P|$ the order of $\ker(\overline{\eta})$ is coprime to p. We have the following:

Lemma 5.5. *Let (P, s) be a pair with $C_{\langle s\rangle}(P) = 1$ and set $G := \langle Ps\rangle$. Let H be a finite group. The following statements are equivalent:*

(i) $\mathbb{F}T^{\Delta}(H, G) \otimes_{kG} F_{P,s}^{G} \neq 0$.

(ii) There exists a pair (P', s') contained in H such that the pair (P, s) is isomorphic to a p'-quotient of the pair (P', s'), that is, there exists a normal p'-subgroup K of $\langle P's'\rangle$ such that $(P, s) \simeq (P'K/K, s'K)$.

Proof. (i) $\Rightarrow$ (ii) Suppose we have $\mathbb{F}T^{\Delta}(H, G) \otimes_{kG} F_{P,s}^{G} \neq 0$. Then there exists a pair (P', s') in H such that

$$F_{P',s'}^{H} \in \mathbb{F}T^{\Delta}(H, G) \otimes_{kG} F_{P,s}^{G}.$$

Via the restriction map this implies that we have

$$F_{P',s'}^{\langle P's'\rangle} \in \mathbb{F}T^{\Delta}\big(\langle P's'\rangle, G\big) \otimes_{kG} F_{P,s}^{G}.$$

Therefore by the argument above we have an isomorphism $(P'K/K, s'K) \simeq (P, s)$ of pairs where K is a normal p'-subgroup of $\langle P's'\rangle$.
(ii) $\Rightarrow$ (i) Suppose $\Phi : (P'K/K, s'K) \to (P, s)$ is an isomorphism of pairs where K is a normal p'-subgroup of $\langle P's'\rangle$. Then we have

$$\mathrm{Ind}_{\langle P's'\rangle}^{H} \mathrm{Inf}_{\langle P's'\rangle/K}^{\langle P's'\rangle} \mathrm{Iso}(\Phi) F_{P,s}^{G} \neq 0.$$

This shows (i). □

Proposition 5.6. *Let (P, s) be a pair. The following are equivalent:*

(i) (P, s) is a D^{Δ}-pair, that is, for any nontrivial normal p'-subgroup N of $\langle Ps\rangle$, we have $\mathrm{Def}_{\langle Ps\rangle/N}^{\langle Ps\rangle} F_{P,s}^{\langle Ps\rangle} = 0$.

(ii) For any finite group H with $|H| < |\langle Ps\rangle|$, we have $\mathbf{e}_{P,s}(H) = \{0\}$.

(iii) *If H is a finite group with $\mathbf{e}_{P,s}(H) \neq \{0\}$, then the pair (P,s) is isomorphic to a p'-quotient of a pair (P',s') contained in H.*

(iv) *The group $\langle Ps\rangle$ does not have any nontrivial normal p'-subgroup.*

(v) *We have $C_{\langle s\rangle}(P) = 1$.*

Proof. (v)⇔(iv)⇔(i) : This follows from Lemma 4.5.
(i)⇒ (iii): Since (P,s) is a D^Δ-pair, we have $C_{\langle s\rangle}(P) = 1$. So (iii) follows from Lemma 5.5.
(iii)⇒ (ii): Assume that (iii) holds and $\mathbf{e}_{P,s}(H) \neq 0$ where H is a finite group with $|H| < |\langle Ps\rangle|$. Then by the assumption, we have $|H| \geq |\langle P's'\rangle| \geq |\langle Ps\rangle|$. Contradiction.
(ii)⇒ (i): Clear. □

Proposition 5.7. *Let (P,s) and (Q,t) be two pairs.*

(i) *If (Q,t) is isomorphic to a p'-quotient of (P,s), then $\mathbf{e}_{P,s} = \mathbf{e}_{Q,t}$.*

(ii) *If (Q,t) is a D^Δ-pair, and if $\mathbf{e}_{P,s} \leqslant \mathbf{e}_{Q,t}$, then (Q,t) is isomorphic to a p'-quotient of (P,s).*

Proof. (i) Assume we have an isomorphism $\phi : (PK/K, sK) \to (Q,t)$ of pairs for some normal p'-subgroup K of $\langle Ps\rangle$. Then

$$F_{P,s}^{\langle Ps\rangle} \otimes_k \mathrm{Inf}_{\langle Ps\rangle/K}^{\langle Ps\rangle} \mathrm{Iso}(\phi^{-1}) F_{Q,t}^{\langle Qt\rangle} \neq 0.$$

Therefore $F_{P,s}^{\langle Ps\rangle} \in \mathbf{e}_{Q,t}\big(\langle Ps\rangle\big)$ which implies that $\mathbf{e}_{P,s} \leqslant \mathbf{e}_{Q,t}$.

Now we also have

$$F_{Q,t}^{\langle Qt\rangle} \otimes_k \mathrm{Iso}(\phi) \mathrm{Def}_{\langle Ps\rangle/K}^{\langle Ps\rangle} F_{P,s}^{\langle Ps\rangle} \neq 0$$

which implies that $F_{Q,t}^{\langle Qt\rangle} \in \mathbf{e}_{P,s}\big(\langle Qt\rangle\big)$. Therefore $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}$ and so $\mathbf{e}_{Q,t} = \mathbf{e}_{P,s}$ as desired.
(ii) Since $\mathbf{e}_{P,s} \leqslant \mathbf{e}_{Q,t}$, we have $F_{P,s}^{\langle Ps\rangle} \in \mathbf{e}_{Q,t}\big(\langle Ps\rangle\big)$. Since (Q,t) is a D^Δ-pair, by the proof of Lemma 5.5, there exists a normal p'-subgroup K of $\langle Ps\rangle$ such that $(Q,t) \simeq (PK/K, sK)$. □

Proposition 5.8. *Let F be a nonzero subfunctor of $\mathbb{F}T^{\Delta}$. If H is a minimal group of F, then $H = \langle Qt \rangle$ for some D^{Δ}-pair (Q,t). Moreover*

$$F(H) \leqslant \bigoplus_{\substack{(Q',t'),D^{\Delta}-pair\\ \langle Q't' \rangle = H}} \mathbb{F}F^{H}_{Q',t'}$$

and $\mathbf{e}_{Q,t} \leqslant F$.

In particular, if $F = \mathbf{e}_{Q,t}$ for some D^{Δ}-pair (Q,t), then

$$\mathbf{e}_{Q,t}\big(\langle Qt \rangle\big) = \bigoplus_{\substack{(Q',t') \simeq (Q,t)\\ \langle Q't' \rangle = \langle Qt \rangle}} \mathbb{F}F^{H}_{Q',t'}.$$

Proof. Let F be a nonzero subfunctor of $\mathbb{F}T^{\Delta}$ and assume H is a minimal group of F. Since $F(H) \neq 0$, there exists a pair $(Q,t) \in \mathcal{Q}_{H,p}$ such that $F^{H}_{Q,t} \in F(H)$. This implies, via the restriction map, that $F^{\langle Qt \rangle}_{Q,t} \in F\big(\langle Qt \rangle\big)$. Since H is a minimal group, this implies that $H = \langle Qt \rangle$. Now if N is a normal p'-subgroup of $\langle Qt \rangle$, then $\mathrm{Def}^{\langle Qt \rangle}_{\langle Qt \rangle/N} F^{\langle Qt \rangle}_{Q,t} = \frac{1}{|N|} F^{\langle Qt \rangle/N}_{QN/N,tN} \neq 0$. Again since H is a minimal group this means that N is trivial and hence the pair (Q,t) is a D^{Δ}-pair. It follows moreover that

$$F(H) \leqslant \bigoplus_{\substack{(Q',t'),D^{\Delta}-pair\\ \langle Q't' \rangle = H}} \mathbb{F}F^{H}_{Q',t'}.$$

For the last part, consider the functor $\mathbf{e}_{Q,t}$ for some D^{Δ}-pair (Q,t). If $F^{\langle Qt \rangle}_{Q',t'} \in \mathbf{e}_{Q,t}\big(\langle Qt \rangle\big)$ for some D^{Δ}-pair (Q',t'), then by the second part of Proposition 5.7, the pair (Q,t) is isomorphic to a p'-quotient of the pair (Q',t'). But the pair (Q',t') is contained in $\langle Qt \rangle$. Thus $(Q',t') \simeq (Q,t)$.
Conversely, if the pairs (Q',t') and (Q,t) are isomorphic via a map ϕ, then we have $F^{\langle Qt \rangle}_{Q',t'} = \mathrm{Iso}(\phi) F^{\langle Qt \rangle}_{Q,t}$. Therefore

$$\mathbf{e}_{Q,t}\big(\langle Qt \rangle\big) = \bigoplus_{\substack{(Q',t') \simeq (Q,t)\\ \langle Q't' \rangle = \langle Qt \rangle}} \mathbb{F}F^{H}_{Q',t'}.$$

□

Let (P,s) be a pair and N a normal p'-subgroup of $\langle Ps \rangle$. Then the pair $(PN/N, sN)$ is a p'-quotient of the pair (P,s) and so by Proposition 5.7 we have $\mathbf{e}_{P,s} = \mathbf{e}_{PN/N,sN}$.

Proposition 5.9. *Let (P,s) be a pair. Then the group $\langle Ps\rangle/C_{\langle s\rangle}(P)$ is the unique, up to isomorphism, minimal group of the functor $\mathbf{e}_{P,s}$. Moreover there is a unique isomorphism class of D^Δ-pairs (P',s') such that $\langle P's'\rangle \cong \langle Ps\rangle/C_{\langle s\rangle}(P)$ and we have $\mathbf{e}_{P',s'} = \mathbf{e}_{P,s}$. Furthermore we have $(P',s') \simeq \big(PC_{\langle s\rangle}(P)/C_{\langle s\rangle}(P), sC_{\langle s\rangle}(P)\big)$.*

Proof. Let (P',s') be a D^Δ-pair such that $\langle P's'\rangle$ is a minimal group of the functor $\mathbf{e}_{P,s}$. By Proposition 5.8, we have $\mathbf{e}_{P',s'} \leqslant \mathbf{e}_{P,s}$. Let $N := C_{\langle s\rangle}(P)$. Then the pair $(PN/N, sN)$ is a D^Δ-pair, and we have $\mathbf{e}_{P,s} = \mathbf{e}_{PN/N,sN}$. Since $(PN/N,sN)$ is a D^Δ-pair, by Proposition 5.7 there exists a normal p'-subgroup K of $\langle P's'\rangle$ such that $(P'K/K, s'K) \simeq (PN/N, sN)$. This means that the idempotent $F^{\langle P's'\rangle/K}_{P'K/K,s'K}$ is in the evaluation at $\langle P's'\rangle/K$ of the functor $\mathbf{e}_{PN/N,sN} = \mathbf{e}_{P,s}$. Since the group $\langle P's'\rangle$ is a minimal group of $\mathbf{e}_{P,s}$ it follows that we must have $K = 1$. Thus we have $(P',s') \simeq (PN/N, sN)$. Therefore we have $\mathbf{e}_{P',s'} = \mathbf{e}_{PN/N,sN} = \mathbf{e}_{P,s}$.

Now we will show the uniqueness of the isomorphism class of the minimal groups of $\mathbf{e}_{P,s}$. Let H be a minimal group of $\mathbf{e}_{P,s}$. It suffices to show that H is isomorphic to $\langle P's'\rangle$. By Proposition 5.8 the group H is of the form $H = \langle Qt\rangle$ for some D^Δ-pair (Q,t). By the first part of the proof we have $\mathbf{e}_{Q,t} = \mathbf{e}_{P,s} = \mathbf{e}_{P',s'}$. Since both (Q,t) and (P,s) are D^Δ-pairs, the equality $\mathbf{e}_{Q,t} = \mathbf{e}_{P',s'}$ implies that (Q,t) is isomorphic to a p'-quotient of (P,s), and vice versa. Therefore we have $(Q,t) \simeq (P',s')$ which implies that $H = \langle Qt\rangle \cong \langle P's'\rangle$ as desired. □

For any pair (P,s) we denote by $(\tilde{P},\tilde{s})$ a representative of the isomorphism class of the pair $(PC_{\langle s\rangle}(P)/C_{\langle s\rangle}(P), sC_{\langle s\rangle}(P))$.

Theorem 5.10. *Let (P,s) be a pair.*

(i) If (Q,t) is isomorphic to a p'-quotient of (P,s) and if (Q,t) is a D^Δ-pair, then (Q,t) is isomorphic to the pair $(\tilde{P},\tilde{s})$. In particular, for any normal p'-subgroup $N \trianglelefteq \langle Ps\rangle$, we have $(PN/N,sN) \simeq (\tilde{P},\tilde{s})$ if and only if $(PN/N,sN)$ is a D^Δ-pair.

(ii) Let $N \trianglelefteq \langle Ps\rangle$ be a normal p'-subgroup. Then the pair $(\tilde{P},\tilde{s})$ is isomorphic to a p'-quotient of $(PN/N,sN)$ and we have $(\tilde{P},\tilde{s}) \simeq (\widetilde{PN/N}, \widetilde{sN})$.

Proof. (i) Since the pair (Q,t) is isomorphic to a p'-quotient of the pair (P,s), by Proposition 5.7, we have $\mathbf{e}_{\tilde{P},\tilde{s}} = \mathbf{e}_{P,s} \leqslant \mathbf{e}_{Q,t}$. Since (Q,t) is a D^Δ-pair, again by Proposition 5.7, the pair (Q,t) is isomorphic to a p'-quotient of $(\tilde{P},\tilde{s})$. But since the pair $(\tilde{P},\tilde{s})$ is a D^Δ-pair, it follows that the pair (Q,t) is isomorphic to the pair $(\tilde{P},\tilde{s})$.

(ii) Since the constant $m_{P,s,N}$ is non-zero, we have $F^{\langle Ps\rangle/N}_{PN/N,sN} \in \mathbf{e}_{P,s}\big(\langle Ps\rangle/N\big) = \mathbf{e}_{\tilde{P},\tilde{s}}\big(\langle Ps\rangle/N\big)$. Therefore we have $\mathbf{e}_{PN/N,sN} \leqslant \mathbf{e}_{\tilde{P},\tilde{s}}$ and since $(\tilde{P},\tilde{s})$ is a D^{Δ}-pair, by Proposition 5.7, $(\tilde{P},\tilde{s})$ is isomorphic to a p'-quotient of $(PN/N, sN)$. Again since the pair $(\tilde{P},\tilde{s})$ is a D^{Δ}-pair, by part (i), it is isomorphic to the pair $(\widetilde{PN/N}, \widetilde{sN})$. □

Let $[D^{\Delta}\text{-}pair]$ denote a set of isomorphism classes of D^{Δ}-pairs. Then the subfunctor lattice of the functor $\mathbb{F}T^{\Delta}$ is isomorphic to the lattice of subsets of the set $[D^{\Delta}\text{-}pair]$ ordered by inclusion.

Theorem 5.11. *Let $\mathcal{S}$ be the lattice of subfunctors of $\mathbb{F}T^{\Delta}$ ordered by inclusion of subfunctors. Let $\mathcal{T}$ be the lattice of subsets of $[D^{\Delta}\text{-}pair]$ ordered by inclusion of subsets. Then the map*

$$\Theta : \mathcal{S} \to \mathcal{T}$$

that sends a subfunctor F to the set $\{(P,s) \in [D^{\Delta}\text{-}pair] : \mathbf{e}_{P,s} \leqslant F\}$, is an isomorphism of lattices with inverse

$$\Psi : \mathcal{T} \to \mathcal{S}$$

that sends a subset A to the functor $\sum_{(P,s)\in A} \mathbf{e}_{P,s}$.

Proof. We need to show that the maps Θ and Ψ are inverse of each other. Let $F \in \mathcal{S}$ be a subfunctor. By Proposition 5.3 we have

$$F = \sum_{\substack{(P,s)\in\Gamma \\ \mathbf{e}_{P,s}\leqslant F}} \mathbf{e}_{P,s}$$

where Γ is a set of representatives of the isomorphism classes of pairs. But for any pair (P,s) we have $\mathbf{e}_{P,s} = \mathbf{e}_{\tilde{P},\tilde{s}}$ and $(\tilde{P},\tilde{s})$ is a D^{Δ}-pair. Therefore we have

$$F = \sum_{\substack{(P,s)\in[D^{\Delta}\text{-}pair] \\ \mathbf{e}_{P,s}\leqslant F}} \mathbf{e}_{P,s}.$$

This shows that $\Psi(\Theta(F)) = F$.

Now let $A \in \mathcal{T}$ be a subset and let $(Q,t) \in \Theta(\Psi(A))$ be a D^{Δ}-pair. Then we have $\mathbf{e}_{Q,t} \leqslant \sum_{(P,s)\in A} \mathbf{e}_{P,s}$ and so by Proposition 5.4 this implies that we have $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}$ for some $(P,s) \in A$. Since both (P,s) and (Q,t) are D^{Δ}-pairs, it follows that $(P,s) \simeq (Q,t)$ and hence $(Q,t) \in A$. This shows that $\Theta(\Psi(A)) \subseteq A$. The inclusion $A \subseteq \Theta(\Psi(A))$ is trivial. Therefore we have $\Theta(\Psi(A)) = A$. □

The following corollary follows immediately from Theorem 5.11.

Corollary 5.12. *We have* $\mathbb{F}T^\Delta = \bigoplus_{(P,s)\in[D^\Delta\text{-}pair]} \mathbf{e}_{P,s}$.

The first statement of Proposition 5.8 can also be made stronger.

Corollary 5.13. *Let* F *be a nonzero subfunctor of* $\mathbb{F}T^\Delta$. *If* H *is a minimal group of* F, *then* $H = \langle Qt\rangle$ *for some* D^Δ*-pair* (Q,t) *and we have*

$$F(H) = \bigoplus_{\substack{(Q',t')\simeq(Q,t)\\ \langle Q't'\rangle=\langle Qt\rangle}} \mathbb{F}F^H_{Q',t'}.$$

Proof. Since H is a minimal group of F, by Proposition 5.8 it follows that $H = \langle Qt\rangle$ for some D^Δ-pair with the property that $\mathbf{e}_{Q,t} \leqslant F$. By Theorem 5.11 we have

$$F = \sum_{\substack{(Q,t)\in[D^\Delta\text{-}pair]\\ \mathbf{e}_{Q,t}\leqslant F}} \mathbf{e}_{Q,t}.$$

Therefore by Proposition 5.8 again we have

$$F(H) = \mathbf{e}_{Q,t}(H) = \bigoplus_{\substack{(Q',t')\simeq(Q,t)\\ \langle Q't'\rangle=\langle Qt\rangle}} \mathbb{F}F^H_{Q',t'}$$

as desired. □

Theorem 5.14. *(i) Let* (P,s) *be a* D^Δ*-pair. Then the subfunctor* $\mathbf{e}_{P,s}$ *of* $\mathbb{F}T^\Delta$ *is isomorphic to the simple functor* $S_{\langle Ps\rangle,W_{P,s}}$ *where* $W_{P,s} = \oplus_{\substack{(Q,t)\simeq(P,s)\\ \langle Qt\rangle=\langle Ps\rangle}}\mathbb{F}F^{\langle Ps\rangle}_{P,s}$.

(ii) Let (P,s) *and* (Q,t) *be* D^Δ*-pairs. Then the functor* $\mathbf{e}_{P,s}$ *and* $\mathbf{e}_{Q,t}$ *are isomorphic if and only if the pairs* (P,s) *and* (Q,t) *are isomorphic, that is, if and only if* $\mathbf{e}_{P,s} = \mathbf{e}_{Q,t}$ *as subfunctors of* $\mathbb{F}T^\Delta$.

(iii) The functor $\mathbb{F}T^\Delta$ *is semisimple. More precisely*

$$\mathbb{F}T^\Delta \cong \bigoplus_{(P,s)} S_{\langle Ps\rangle,W_{P,s}},$$

where (P,s) *runs through a set of representatives of isomorphism classes of* D^Δ*-pairs.*

Proof. (i) By Theorem 5.11 the lattice of subfunctors of $\mathbf{e}_{P,s}$ is isomorphic to the lattice of subsets of the set $\Theta(\mathbf{e}_{P,s}) = \{(Q,t) \in [D^\Delta\text{-}pair] : \mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}\} = \{(P,s)\}$. Therefore the subfunctor $\mathbf{e}_{P,s}$ is simple. By Proposition 5.9 the group $\langle Ps\rangle$ is a minimal group of the functor $\mathbf{e}_{P,s}$. By Proposition 5.8 we have $\mathbf{e}_{P,s}\big(\langle Ps\rangle\big) = W_{P,s}$. Moreover, by [7, Theorem 4.2.5], the module $W_{P,s}$ is a simple module for the essential algebra $\mathcal{E}^\Delta\big(\langle Ps\rangle\big)$. Thus we have $\mathbf{e}_{P,s} \simeq S_{\langle Ps\rangle, W_{P,s}}$ as desired.

(ii) Clearly if $(P,s) \simeq (Q,t)$, then $\mathbf{e}_{P,s} = \mathbf{e}_{Q,t}$ as subfunctors of $\mathbb{F}T^\Delta$. In particular $\mathbf{e}_{P,s} \cong \mathbf{e}_{Q,t}$. Conversely, if $\mathbf{e}_{P,s} \cong \mathbf{e}_{Q,t}$, then $\mathbf{e}_{P,s}$ and $\mathbf{e}_{Q,t}$ have the same minimal groups, so $(P,s) \simeq (Q,t)$ by Proposition 5.9.

(iii) By Assertion (i), this is just a reformulation of Corollary 5.12. □

Proposition 5.15. *Let (P,s) be a pair. Then for any finite group H, the $\mathbb{F}$-vector space $\mathbf{e}_{P,s}(H)$ is the subspace of $\mathbb{F}T(H)$ generated by the set of primitive idempotents $F^H_{Q,t}$ where (Q,t) runs over a set of conjugacy classes of pairs in H with the property that (P,s) is isomorphic to a p'-quotient of (Q,t).*

Proof. Since the pair $(\tilde{P},\tilde{s})$ is isomorphic to a p'-quotient of the pair (P,s) and since $\mathbf{e}_{P,s} = \mathbf{e}_{\tilde{P},\tilde{s}}$, we may assume that the pair (P,s) is a D^Δ-pair. Since $\mathbf{e}_{P,s}(H)$ is an ideal of $\mathbb{F}T(H)$, it has a $\mathbb{F}$-basis consisting of a set of primitive idempotents $F^H_{Q,t}$. If $F^H_{Q,t} \in \mathbf{e}_{P,s}(H)$, then $F^{\langle Qt\rangle}_{Q,t} \in \mathbf{e}_{P,s}\big(\langle Qt\rangle\big)$ and so $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}$. Since (P,s) is a D^Δ-pair, by Proposition 5.7, it is isomorphic to a p'-quotient of the pair (Q,t). Conversely, if (P,s) is isomorphic to a p'-quotient of the pair (Q,t), then again by Proposition 5.7, we have $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}$. So we have $F^{\langle Qt\rangle}_{Q,t} \in \mathbf{e}_{P,s}\big(\langle Qt\rangle\big)$ and hence $F^H_{Q,t} \in \mathbf{e}_{P,s}(H)$. The result follows. □

Theorem 5.16. *Let (P,s) be a D^Δ-pair. Then for any finite group H, the $\mathbb{F}$-dimension of $S_{\langle Ps\rangle, W_{P,s}}(H)$ is equal to the number of conjugacy classes of pairs (Q,t) in H such that $(\tilde{Q},\tilde{t}) \simeq (P,s)$.*

Proof. By Proposition 5.15, $\mathbf{e}_{P,s}(H)$ is generated by the idempotents $F^H_{Q,t}$ where (Q,t) is a pair in H with the property that the pair $(\tilde{P},\tilde{s}) \simeq (P,s)$ is isomorphic to a p'-quotient of the pair (Q,t). Since (P,s) is a D^Δ-pair, Theorem 5.10 implies that $(\tilde{Q},\tilde{t}) \simeq (P,s)$. The result follows. □

Corollary 5.17. *Let H be a finite group. The $\mathbb{F}$-dimension of $S_{1,\mathbb{F}}(H)$ is equal to the number of isomorphism classes of simple kH-modules.*

Proof. By Theorem 5.16, $\dim_{\mathbb{F}} S_{1,\mathbb{F}}(H)$ is equal to the number of conjugacy classes of pairs (Q,t) in H such that $(\tilde{Q},\tilde{t}) \simeq (1,1)$. Suppose (Q,t) is a pair with $(\tilde{Q},\tilde{t}) \simeq (1,1)$. Then we have $\tilde{Q} = 1$ and $\tilde{t} = 1$. So there exists a normal p'-subgroup N of $\langle Qt \rangle$ such that $(QN/N, tN) \simeq (1,1)$. Since $|Q|$ and $|N|$ are coprime, this implies that $Q = 1$. We also have $t \in N$. But then $N \trianglelefteq \langle t \rangle$ implies that $N = \langle t \rangle$. Therefore the number of conjugacy classes of pairs (Q,t) in H such that $(\tilde{Q},\tilde{t}) \simeq (1,1)$ is equal to the number of conjugacy classes of p'-elements in H. The result follows. □

Theorem 5.18. *The functor $S_{1,\mathbb{F}}$ is isomorphic to the functor that sends a finite group H to the subspace $\mathbb{F}K_0(kH)$ of $\mathbb{F}T^{\Delta}(H)$ generated by the projective indecomposable kH-modules.*

Proof. Let H be a finite group. We have

$$S_{1,\mathbb{F}}(H) = (\mathbb{F}T^{\Delta}(H,1) \otimes_{\mathbb{F}} \mathbb{F})/J_{1,\mathbb{F}}(H) \cong \mathbb{F}T^{\Delta}(H,1)/J_{1,\mathbb{F}}(H)$$

where $J_{1,\mathbb{F}}(H) = \{\phi \in \mathbb{F}T^{\Delta}(H,1) : \forall \psi \in \mathbb{F}T^{\Delta}(1,H), (\psi \circ \phi) \cdot 1 = 0\}$. Now $\mathbb{F}T^{\Delta}(H,1)$ is isomorphic to the subspace $\mathbb{F}K_0(kH)$ of $\mathbb{F}T(H)$ generated by the isomorphism classes of projective indecomposable kH-modules. Similarly any $W \in \mathbb{F}T^{\Delta}(1,H)$ can be identified with $W^* \in \mathbb{F}K_0(kH)$. As in [8] we have the following:
For any p-permutation kH-modules V and W we have

$$(W^* \otimes_{kH} V) \cdot 1 = \dim_k(W^* \otimes_{kH} V) = \dim_k(\mathrm{Hom}_{kH}(W,V)).$$

Therefore $J_{1,\mathbb{F}}(H)$ is the right kernel of the bilinear form

$$< -, - >: \mathbb{F}K_0(kH) \to \mathbb{F}$$

defined as $< W, V > := \dim_k(\mathrm{Hom}_{kH}(W,V))$. But the matrix that represents this bilinear form is the Cartan matrix of kH. Since the Cartan matrix of a group algebra is non-degenerate, it follows that $J_{1,\mathbb{F}}(H) = 0$. Therefore we have

$$S_{1,\mathbb{F}}(H) = \mathbb{F}T^{\Delta}(H,1) \otimes_{\mathbb{F}} \mathbb{F} \cong \mathbb{F}T^{\Delta}(H,1) \cong \mathbb{F}K_0(kH).$$

Note that both of these isomorphisms are functorial in H. The result follows. □

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