Galois descent of equivalences between blocks of p-nilpotent groups *

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Abstract

We give sufficient conditions on p-blocks of p-nilpotent groups over $\mathbb{F}_p$ to be splendidly Rickard equivalent and p-permutation equivalent to their Brauer correspondents. The paper also contains Galois descent results on p-permutation modules and p-permutation equivalences that hold for arbitrary groups.

1 Introduction

In [KL18], Kessar and Linckelmann proved that Broué's Abelian Defect Group Conjecture (originally stated over splitting fields) holds for blocks with cyclic defect groups over *arbitrary* fields of characteristic $p > 0$, in particular over the prime field $\mathbb{F}_p$. More precisely, if G is a finite group and b is a block idempotent of $\mathbb{F}_pG$ with cyclic defect group D, then there exists a *splendid Rickard equivalence* between $\mathbb{F}_pGb$ and its Brauer correspondent block algebra $\mathbb{F}_pH\mathrm{Br}_D(b)$, where $H = N_G(D)$ and $\mathrm{Br}_D\colon (\mathbb{F}_pG)^D \to \mathbb{F}_pC_G(D)$ is the Brauer homomorphism, an $\mathbb{F}_p$-algebra homomorphism which is given by truncation.

In this paper we investigate if a similar phenomenon holds for blocks of p-nilpotent groups. In this case, a positive answer over a splitting field F of characteristic $p > 0$ of G was given by Rickard, see [R96], even without the assumption of abelian defect groups. There, he introduced and used the notion of an *endosplit p-permutation resolution* in order to construct such splendid Rickard equivalences. We have two main results. The first gives sufficient conditions under which there exists such a splendid Rickard equivalence between Brauer corresponding blocks of a p-nilpotent group over $\mathbb{F}_p$. The second gives sufficient conditions under which the weaker form of equivalence, namely a *p-permutation equivalence*, exists over $\mathbb{F}_p$.

So let G be a *p-nilpotent group*, i.e., a finite group whose largest normal p'-subgroup N is a complement to a (and then each) Sylow p-subgroup of G. Moreover, let F be a finite splitting field of G and its subgroups of characteristic $p > 0$. Let $\tilde{b}$ be a block idempotent of $\mathbb{F}_pG$ and

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let b be a block idempotent of FG which occurs in a primitive decomposition of $\tilde{b}$ in $Z(FG)$. Then b is contained in $Z(FN)$. Let e be a block idempotent of FN that occurs in the primitive decomposition of b in $Z(FN)$. Adjoining the coefficients of e and of b to $\mathbb{F}_p$, one obtains subfields $\mathbb{F}_p[b] \subseteq \mathbb{F}_p[e] \subseteq F$, since b is a sum of G-conjugates of e.

Theorem A *Let G be a p-nilpotent group and let $\tilde{b}$ be a block idempotent of $\mathbb{F}_pG$. Suppose that p is odd or $\tilde{b}$ has abelian defect groups, and suppose that, with the above notation, $\mathbb{F}_p[b] = \mathbb{F}_p[e]$. Then there exists a splendid Rickard equivalence between the block algebra $\mathbb{F}_pG\tilde{b}$ and its Brauer correspondent block algebra.*

Theorem A follows from the more precise statement in Proposition 5.8 and Remark 5.9. The proof uses Rickard's original approach in [R96] involving endosplit p-permutation resolutions, a descent result in [KL18], and the classification of endopermutation modules over p-groups, see [T07] for a survey article on the latter.

There are weaker forms of equivalences between blocks than splendid Rickard equivalences, as for instance p-permutation equivalences which were introduced in [BX08] and extended in [L09] and [BP20]. See Section 4 for a definition.

Theorem B *Let G be a p-nilpotent group with abelian Sylow p-subgroup and let $\tilde{b}$ be a block idempotent of $\mathbb{F}_pG$. Then there exists a p-permutation equivalence between $\mathbb{F}_pG\tilde{b}$ and its Brauer correspondent block algebra.*

Theorem B follows from the more precise statement in Corollary 5.15. The proof uses again Rickard's construction and Galois descent arguments for the representation ring of trivial source modules developed in this paper, see Theorem 2.6 and Lemma 4.3. The reason that we only obtain a p-permutation equivalence and not a splendid Rickard equivalence in Theorem B, is that we don't have a descent result analogous to Lemma 4.3 for splendid Rickard equivalences and that the descent result from [KL18] cannot be applied without the assumption that $\mathbb{F}_p[b] = \mathbb{F}_p[e]$, see also Remark 5.16. Because Theorem 2.6 and Lemma 4.3 are of independent interest we include them in the introduction as Theorems C and D. For these two results, G and H can be arbitrary finite groups and we assume that F is a splitting field for G and H and their subgroups.

Theorem C *Let F' be a subfield of F and $\Delta := \mathrm{Gal}(F/F')$. Then, scalar extension from F' to F induces an isomorphism $T_{F'}(G) \to T_F(G)^{\Delta}$ from the trivial source ring of $F'G$ to the Δ-fixed points of $T_F(G)$.*

Theorem D *Let b and c be block idempotents of FG and FH, respectively. Let $\tilde{b}$ and $\tilde{c}$ denote the block idempotents of $\mathbb{F}_pG$ and $\mathbb{F}_pH$ associated to b and c, respectively, as in Proposition 4.1(a). Moreover, let $\omega \in T^{\Delta}(FGb, FHc)$ be a p-permutation equivalence between FGb and FHc. Suppose that $\mathrm{stab}_\Gamma(\omega) = \mathrm{stab}_\Gamma(b) = \mathrm{stab}_\Gamma(c)$. Then there exists a p-permutation equivalence between $\mathbb{F}_pG\tilde{b}$ and $\mathbb{F}_pH\tilde{c}$.*

The paper is arranged as follows. In Section 2 we prove Theorem 2.6. The definition and basic properties of endosplit p-permutation resolutions are given in Section 3. In Section 4 we collect basic results on the Galois group action on blocks and prove Lemma 4.3. Finally, in Section 5 we prove Theorems A and B.

Our notation is standard. For any rings R and S we denote by ${}_R\mathsf{mod}$ (resp. ${}_R\mathsf{mod}_S$) the categories of finitely generated left R-modules (resp. (R, S)-bimodules). For objects M and N in a module category or chain complex category we write $M \mid N$ to indicate that M is isomorphic to a direct summand of N. If H and K are subgroups of a finite group G, then $g \in G/H$ (resp. $g \in H \setminus G/K$) indicates that g runs through a set of representatives of the given cosets (resp. double cosets).

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2 Galois descent of p-permutation modules

Throughout this paper, G and H denote finite groups and F a finite field of characteristic p which is a splitting field for all subgroups of H and G. Moreover, $\Gamma := \mathrm{Gal}(F/\mathbb{F}_p)$ denotes the Galois group of F over $\mathbb{F}_p$. For any subfield $F' \subseteq F$ one has functors

$$-_{F'}\colon {}_{FG}\mathsf{mod} \to {}_{F'G}\mathsf{mod} \quad \text{and} \quad F \otimes_{F'} -\colon {}_{F'G}\mathsf{mod} \to {}_{FG}\mathsf{mod}$$

defined by restriction and extension of scalars.

2.1 Proposition *Let Q be a p-subgroup of G and let F' be a subfield of F. If $M \in {}_{F'G}\mathsf{mod}$ is relatively Q-projective then $F \otimes_{F'} M \in {}_{FG}\mathsf{mod}$ is relatively Q-projective. If $N \in {}_{FG}\mathsf{mod}$ is relatively Q-projective then $N_{F'} \in {}_{F'G}\mathsf{mod}$ is relatively Q-projective.*

Proof This follows immediately from the fact that restriction and extension of scalars commute with Ind_Q^G. ∎

For each $\sigma \in \Gamma$ one has a functor

$$^{\sigma}-\colon {}_{FG}\mathsf{mod} \to {}_{FG}\mathsf{mod} \tag{1}$$

which assigns to $M \in {}_{FG}\mathsf{mod}$ the FG-module ${}^{\sigma}M$ whose underlying abelian group is equal to M and whose FG-module structure is given by restriction along the ring isomorphism $\sigma^{-1}\colon FG \to FG$, $\sum_{g\in G} \alpha_g g \mapsto \sum_{g\in G} \sigma^{-1}(\alpha_g) g$. For any FG-module homomorphism f one has ${}^{\sigma}f = f$. Similarly one defines the functor ${}^{\sigma}-\colon {}_{FG}\mathsf{mod}_{FH} \to {}_{FG}\mathsf{mod}_{FH}$. For any $M \in {}_{FG}\mathsf{mod}$ we set $\mathrm{stab}_{\Gamma}(M) := \{\sigma \in \Gamma \mid {}^{\sigma}M \cong M\}$.

We recall from [L18a, Definition 5.4.10] the definition of the Brauer construction functor

$$-(P)\colon {}_{FG}\mathsf{mod} \to {}_{F[N_G(P)/P]}\mathsf{mod}\,,$$

for any p-subgroup P of G, and we denote by

$$-^{\circ} := \mathrm{Hom}_F(-, F)\colon {}_{FG}\mathsf{mod} \to {}_{FG}\mathsf{mod}$$

the functor of taking F-duals. The above functors extend to functors between appropriate categories of (co-)chain complexes and they have the following properties.

2.2 Lemma *Let G, H and K be finite groups. Further, let M and N be FG-modules, U an (FG, FH)-bimodule, V an (FH, FK)-bimodule, $L \leqslant G$ a subgroup, $P \leqslant G$ a p-subgroup and $\sigma, \tau \in \Gamma$. Moreover, let $F' \subseteq F$ be a subfield and set $\Delta := \mathrm{Gal}(F/F')$. Then one has*

(a) ${}^{\tau\circ\sigma}M = {}^{\tau}({}^{\sigma}M)$, ${}^{\sigma}(M \oplus N) = {}^{\sigma}M \oplus {}^{\sigma}N$, *and* ${}^{\sigma}(M \otimes_F N) = {}^{\sigma}M \otimes_F {}^{\sigma}N$.

(b) $\mathrm{Res}_L^G({}^{\sigma}M) = {}^{\sigma}\big(\mathrm{Res}_L^G M\big)$ *and* $\mathrm{Ind}_L^G({}^{\sigma}M) = {}^{\sigma}\Big(\mathrm{Ind}_L^G M\Big)$.

(c) $({}^{\sigma}M)^{\circ} = {}^{\sigma}(M^{\circ})$, ${}^{\sigma}(U \otimes_{FH} V) = {}^{\sigma}U \otimes_{FH} {}^{\sigma}V$, *and* $({}^{\sigma}M)(P) = {}^{\sigma}((M)(P))$.

(d) $F \otimes_{F'} M_{F'} \cong \bigoplus_{\sigma\in\Delta} {}^{\sigma}M$.

Proof The proofs of (a)–(c) are straightforward. For a proof of Part (d) see [KL18, Proposition 6.3]. ∎

2.3 Corollary *Let F' be a subfield of F. An indecomposable $F'G$-module M and the indecomposable direct summands of $F \otimes_{F'G} M$ have the same vertices. Similarly, an indecomposable FG-module N and the indecomposable direct summands of $N_{F'}$ have the same vertices.*

Proof This follows from Lemma 2.2(d) and Proposition 2.1. The first statement also follows from [F82, Chapter III, Lemma 4.14]. ∎

Feit attributes the following theorem to Brauer.

2.4 Theorem *[F82, Theorem 19.3] Let F' be a finite field, A a finite dimensional F'-algebra, K/F' a field extension, V an absolutely irreducible $K \otimes_{F'} A$-module such that $\mathrm{tr}_V(a) \in F'$ for every $a \in A$. Then V has an A-form, i.e., there exists an absolutely irreducible A-module W such that $K \otimes_{F'} W \cong V$ as $K \otimes_{F'} A$-modules.*

2.5 Corollary *Let V be an irreducible FG-module and let $F' := F^{\Delta}$ denote the fixed field of $\Delta := \mathrm{stab}_{\Gamma}(V)$. Then there exists an (absolutely) irreducible $F'G$-module W with $V \cong F \otimes_{F'} W$.*

Proof This follows from the above theorem noting that if ${}^{\sigma}V \cong V$ for some $\sigma \in \Gamma$ then $\sigma(\mathrm{tr}_V(g)) = \mathrm{tr}_V(g)$ for all $g \in G$. ∎

Recall from [L18a, Section 5.11] that, for any field F' of characteristic $p > 0$, a *p-permutation $F'G$-module* M is a direct summand of a finitely generated permutation $F'G$-module. Equivalently, the restriction of M to any p-subgroup of G is a permutation module. Also equivalently, the sources of the indecomposable direct summands of M are trivial modules. We denote the Grothendieck group of p-permutation $F'G$-modules V with respect to split short exact sequences by $T_{F'}(G)$. It is a commutative ring with multiplication induced by $-\otimes_{F'} -$. The class of V in $T_{F'}(G)$ is denoted by $[V]$. The classes of indecomposable p-permutation $F'G$-modules form a standard $\mathbb{Z}$-basis of $T_{F'}(G)$. If E is a field extension of F then the ring homomorphism $T_F(G) \to T_E(G)$ of scalar extension is an isomorphism, see [BG07, Theorem 1.9]. The Galois conjugation functors in (1) induce an action of the group Γ on $T_F(G)$ via ring isomorphisms which stabilizes the standard basis. For a subfield F' of F the functors of scalar restriction and extension induce a group homomorphism $T_F(G) \to T_{F'}(G)$ and a ring homomorphism

$$T_{F'}(G) \to T_F(G) \tag{2}$$

which is injective by the Deuring-Noether Theorem and whose image is contained in the subring $T_F(G)^{\Delta}$ of Δ-fixed points, where $\Delta := \mathrm{Gal}(F/F')$. Note that also the abelian group $T_F(G)^{\Delta}$ has a standard $\mathbb{Z}$-basis, namely the Δ-orbit sums of the standard basis of $T_F(G)$. The goal of this section is the following theorem.

2.6 Theorem *Let F' be a subfield of F and $\Delta := \mathrm{Gal}(F/F')$. Then the ring homomorphism $T_{F'}(G) \to T_F(G)^{\Delta}$ in (2) induced by scalar extension is an isomorphism, mapping the standard basis to the standard basis.*

Before proving the above theorem we need the following proposition.

2.7 Proposition *Let V be an indecomposable p-permutation FG-module and let $F' = F^{\Delta}$ be the fixed field of $\Delta := \mathrm{stab}_{\Gamma}(V)$. Then there exists a unique (up to isomorphism) indecomposable p-permutation $F'G$-module W such that $V \cong F \otimes_{F'} W$ as FG-modules.*

Proof Let P be a vertex of the module V. Then the Brauer construction $V(P)$ of V is projective indecomposable as an $F[N_G(P)/P]$-module and its inflation is the Green correspondent of V, see [L18a, Theorem 5.10.5]. The quotient module $S := V(P)/J(V(P))$ is absolutely irreducible since F is a splitting field. Since the Green correspondence and taking projective covers commutes with Galois conjugation, the stabilizer of the isomorphism class of S in Γ is equal to Δ.

By Corollary 2.5, there exists an irreducible $F'[N_G(P)/P]$-module T such that $S \cong F \otimes_{F'} T$. Let W' be a projective indecomposable $F'[N_G(P)/P]$-module such that $W'/J(W') \cong T$, and let $W \in {}_{F'G}\mathsf{mod}$ be the Green correspondent of the inflation of W'. We will show that $V \cong F \otimes_{F'} W$.

First we claim that the projective $F[N_G(P)/P]$-module $F \otimes_{F'} W'$ is a projective cover of S. In fact, $J(F \otimes_{F'} F'[N_G(P)/P]) = F \otimes_{F'} J(F'[N_G(P)/P])$, see [L18a, Propositions 1.16.14 and 1.16.18], so that $J(F \otimes_{F'} W') = F \otimes_{F'} J(W')$. With this we obtain

$$(F \otimes_{F'} W')/J(F \otimes_{F'} W') = (F \otimes_{F'} W')/(F \otimes_{F'} J(W')) \cong F \otimes_{F'} (W'/J(W')) \cong S\,,$$

establishing the claim. Thus, $F \otimes_{F'} W' \cong V(P)$ as $F[N_G(P)/P]$-modules and also as $F[N_G(P)]$-modules after inflation. Since W is the Green correspondent of W', we have

$$F \otimes_{F'} W \mid F \otimes_{F'} \mathrm{Ind}_{N_G(P)}^G(W') \cong \mathrm{Ind}_{N_G(P)}^G(F \otimes_{F'} W') \cong \mathrm{Ind}_{N_G(P)}^G(V(P))\,.$$

But since the modules V and $V(P)$ are Green correspondents, the module V is the unique indecomposable direct summand of $\mathrm{Ind}_{N_G(P)}^G(V(P))$ with vertex P and has multiplicity one in $\mathrm{Ind}_{N_G(P)}^G(V(P))$. Now Corollary 2.3 implies $F \otimes_{F'} W \cong V$, as desired. □

Proof *of Theorem 2.6.* It suffices to show that every standard basis element of $T_F(G)^\Delta$ comes via scalar extension from $T_{F^\Delta}(G)$. So let V be an indecomposable p-permutation FG-module and set $\Delta' := \mathrm{stab}_\Gamma([V])$. Then the Δ-orbit sum of $[V]$, i.e., the class of $\bigoplus_{\sigma\in\Delta/(\Delta\cap\Delta')} {}^\sigma V$ is a standard basis element of $T_F(G)^\Delta$ and every standard basis element is of this form. By Proposition 2.7 there exists an indecomposable p-permutation $F^{\Delta'}G$-module W' such that $V \cong F \otimes_{F^{\Delta'}} W'$. By Lemma 2.2(d), we have

$$\bigoplus_{\sigma\in\Delta/(\Delta\cap\Delta')} {}^\sigma V \cong F \otimes_{F^{\Delta'}} \Big(\bigoplus_{\sigma\in\Delta\Delta'/\Delta'} {}^\sigma W'\Big) \cong F \otimes_{F^{\Delta\Delta'}} (W'_{F^{\Delta\Delta'}}) \cong F \otimes_{F^\Delta} W$$

with $W := F^\Delta \otimes_{F^{\Delta\Delta'}} (W'_{F^{\Delta\Delta'}})$. □

3 Endosplit p-permutation resolutions

In this section, and only this section, F can be any field of characteristic p. The following concept is due to Rickard, see [R96, Section 7].

3.1 Definition Let M be a finitely generated FG-module. An *endosplit p-permutation resolution* of M is a bounded chain complex X of p-permutation FG-modules with homology concentrated in degree 0 such that $H_0(X) \cong M$ and such that $X \otimes_F X^\circ$ is split as chain complex of FG-modules (with G acting diagonally and X° denoting the F-dual of X). Here, X° is again considered as a chain complex.

3.2 Remark Let X be an endosplit p-permutation resolution of a finitely generated FG-module M.

(a) Every direct summand X' of X is again an endosplit p-permutation resolution of $H_0(X')$.

(b) We can decompose X into a direct sum $X = X' \oplus X''$ of chain complexes such that X'' is contractible and X' has no contractible non-zero direct summand. With X, also X' is then an endo-split p-permutation resolution of M. If $X'' = 0$, we say that X is *contractible-free*.

(c) Taking the 0-th homology induces an F-algebra isomorphism

$$\rho : \mathrm{End}_{K({}_{FG}\mathsf{mod})}(X) \cong \mathrm{End}_{FG}(M)\,, \tag{3}$$

where $K({}_{FG}\mathsf{mod})$ denotes the homotopy category of chain complexes in ${}_{FG}\mathsf{mod}$, see [L18b, Proposition 7.11.2]. If $N \mid M$, then the projection map onto N yields an idempotent in $\mathrm{End}_{FG}(M)$ and hence an idempotent in $\mathrm{End}_{K({}_{FG}\mathsf{mod})}(X)$ via the isomorphism in (3). This idempotent lifts to an idempotent α in $\mathrm{End}_{Ch({}_{FG}\mathsf{mod})}(X)$, where $Ch({}_{FG}\mathsf{mod})$ denotes the category of chain complexes in ${}_{FG}\mathsf{mod}$. It follows that the direct summand $\alpha(X)$ of X is an endosplit p-permutation resolution of N. The lifted idempotent is not unique up to conjugation, but $\alpha(X)$ is unique up to isomorphism and contractible direct summands. Therefore, if X is contractible-free, then $\alpha(X)$ is uniquely determined by N up to isomorphism in $Ch({}_{FG}\mathsf{mod})$.

(d) Suppose that $M \cong F \otimes_{F'} M'$ for some subfield $F' \subseteq F$ and some $M' \in {}_{F'G}\mathsf{mod}$. Then, $M_{F'} \cong M'^{[F:F']}$ in ${}_{F'G}\mathsf{mod}$ and $X_{F'} \in Ch({}_{F'G}\mathsf{mod})$ is an endosplit p-permutation resolution of $M'^{[F:F']}$. By Part (c), also M' has an endosplit p-permutation resolution. Conversely, if M' has an endosplit p-permutation resolution $X' \in Ch({}_{F'G}\mathsf{mod})$ then $F \otimes_{F'} X'$ is an endosplit p-permutation resolution of $F \otimes_{F'} M' \cong M$.

3.3 Lemma *Let X_V, X_U, $X_{V'}$ and $X_{U'}$ be endosplit p-permutation resolutions of $V, U, V', U' \in {}_{FG}\mathsf{mod}$, respectively, and assume that X_V and $X_{V'}$ are contractible-free. Suppose further that $X_V \oplus X_U \cong X_{V'} \oplus X_{U'}$ in $Ch({}_{FG}\mathsf{mod})$ are endo-split p-permutation resolutions of $V \oplus U$ and $V' \oplus U'$, respectively, and that $V \cong V'$ in ${}_{FG}\mathsf{mod}$. Then $U \cong U'$ in ${}_{FG}\mathsf{mod}$ and $X_V \cong X_{V'}$ in $Ch({}_{FG}\mathsf{mod})$.*

Proof Taking 0-th homology of $X_V \oplus X_U$ and $X_{V'} \oplus X_{U'}$ yields $V \oplus U \cong V' \oplus U'$, and the Krull-Schmidt Theorem implies $U \cong U'$. For the second statement let $\phi\colon X_V \oplus X_U \to X_{V'} \oplus X_{U'}$ be an isomorphism in $Ch({}_{FG}\mathsf{mod})$. Then $\phi(X_V)$ and $X_{V'}$ are both direct summands of $X_{V'} \oplus X_{U'}$ and contractible-free endo-split p-permutation resolutions of V. Therefore, by [L18b, Proposition 7.11.2] (see also Remark 3.2(b)) they are isomorphic. □

4 Galois descent of p-permutation equivalences

Since the Galois group Γ acts via $\mathbb{F}_p$-algebra automorphisms on the group algebra FG and also on $Z(FG)$, it permutes the block idempotents of FG.

4.1 Proposition *[BKY20, Proposition 4.1]* (a) *Let b be a block idempotent of FG. Then $\tilde{b} := \mathrm{Tr}_\Gamma(b) := \sum_{\sigma \in \Gamma/\mathrm{stab}_\Gamma(b)} {}^{\sigma}b$ is a block idempotent of $\mathbb{F}_pG$.*

(b) *The map $b \mapsto \tilde{b}$ induces a bijection between the set of Γ-orbits of block idempotents of FG and the set of block idempotents of $\mathbb{F}_pG$.*

(c) *If b is a block idempotent of FG and $\tilde{b} := \mathrm{Tr}_\Gamma(b)$ is the block idempotent of $\mathbb{F}_pG$ associated to it as in (a) then $\tilde{b}$ and b have the same defect groups.*

4.2 Lemma *Let b be a block of FG with a defect group P and c be the block of $FN_G(P)$ which is in Brauer correspondence with b. For any $\sigma \in \Gamma$, the blocks ${}^{\sigma}b$ and ${}^{\sigma}c$ are again in Brauer correspondence. In particular, the stabilizers of b and c in Γ are the same. Moreover, the blocks $\tilde{b} = \mathrm{Tr}_{\Gamma}(b)$ of $\mathbb{F}_pG$ and $\tilde{c} = \mathrm{Tr}_{\Gamma}(c)$ of $\mathbb{F}_pN_G(P)$ are Brauer correspondents.*

Proof The first assertion follows immediately from the fact that the action of Γ and the Brauer map Br_P commute. We have $\sigma \in \mathrm{stab}_{\Gamma}(b) \iff \sigma(b) = b \iff \mathrm{Br}_P(\sigma(b)) = \mathrm{Br}_P(b) \iff \sigma(c) = c \iff \sigma \in \mathrm{stab}_{\Gamma}(c)$. The last statement follows easily from the additivity of the Brauer map. □

Let F' be a field of characteristic p and let b and c be central idempotents of $F'G$ and $F'H$, respectively. As usual we identify $F'[G \times H] = F'G \otimes_{F'} F'H$ as F-algebras and we identify $(F'Gb, F'Hc)$-bimodules with left $F'[G \times H](b \otimes c^*)$-modules, where c^* is defined by applying the F'-linear extension of $h \mapsto h^{-1}$ to c. We write $T^{\Delta}(F'Gb, F'Hc)$ for the subgroup of $T_{F'}(G \times H)$ spanned by indecomposable $F'[G \times H](b \otimes c^*)$-modules whose vertices are twisted diagonal, i.e., of the form $\{(\phi(y), y) \mid y \in Q\}$ for some isomorphism $\phi \colon Q \to P$ between p-subgroups P and Q of G and H, respectively. Recall from [BP20] that a *p-permutation equivalence* between $F'Gb$ and $F'Hc$ is an element $\omega \in T^{\Delta}(F'Gb, F'Hc)$ such that $\omega \cdot_H \omega^{\circ} = [F'Gb]$ in $T^{\Delta}(F'Gb, F'Gb)$ and $\omega^{\circ} \cdot_G \omega = [F'Hc]$ in $T^{\Delta}(F'Hc, F'Hc)$. Here, $\cdot_H$ is induced by $- \otimes_{F'H} -$, and $\omega^{\circ} \in T_{F'}(H \times G)$ is given by taking the F'-dual of ω. Note that if $F' = F$ then $\mathrm{stab}_{\Gamma}(\omega) \leqslant \mathrm{stab}_{\Gamma}(b)$ and $\mathrm{stab}_{\Gamma}(\omega) \leqslant \mathrm{stab}_{\Gamma}(c)$.

4.3 Lemma *Let b and c be block idempotents of FG and FH, respectively. Let $\tilde{b}$ and $\tilde{c}$ denote the block idempotents of $\mathbb{F}_pG$ and $\mathbb{F}_pH$ associated to b and c, respectively, as in Proposition 4.1(a). Moreover, let $\omega \in T^{\Delta}(FGb, FHc)$ be a p-permutation equivalence between FGb and FHc. Suppose that we have $\Delta := \mathrm{stab}_{\Gamma}(\omega) = \mathrm{stab}_{\Gamma}(b) = \mathrm{stab}_{\Gamma}(c)$. Then there exists a p-permutation equivalence between $\mathbb{F}_pG\tilde{b}$ and $\mathbb{F}_pH\tilde{c}$.*

Proof For any $\sigma \in \Gamma$, the Galois conjugate ${}^{\sigma}\omega$ is a p-permutation equivalence between $FG\,{}^{\sigma}b$ and $FH\,{}^{\sigma}c$. Hence the sum $\sum_{\sigma \in \Gamma/\Delta} {}^{\sigma}\omega \in T^{\Delta}(FG\tilde{b}, FH\tilde{c})$ is a p-permutation equivalence between $\bigoplus_{\sigma \in \Gamma/\Delta} FG\,{}^{\sigma}b = FG\tilde{b}$ and $\bigoplus_{\sigma \in \Gamma/\Delta} FH\,{}^{\sigma}c = FH\tilde{c}$. Note that the sum $\sum_{\sigma \in \Gamma/\Delta} {}^{\sigma}\omega \in T_F(G \times H)$ is fixed under Γ. By Theorem 2.6, there exists $\tilde{\omega} \in T_{\mathbb{F}_p}(G, H)$ such that $\sum_{\sigma \in \Gamma/\Delta} {}^{\sigma}\omega = F \otimes_{\mathbb{F}_p} \tilde{\omega}$. It follows that $\tilde{\omega}$ is a p-permutation equivalence between $\mathbb{F}_pG\tilde{b}$ and $\mathbb{F}_pH\tilde{c}$. □

5 p-nilpotent groups

Throughout this section we assume that G is a p-nilpotent group. Thus, G has a normal p'-subgroup N such that G/N is a p-group. We fix a block idempotent b of FG and denote by $\tilde{b} := \mathrm{Tr}_{\Gamma}(b)$ the corresponding block idempotent of $\mathbb{F}_pG$, see Proposition 4.1(a). Moreover, we fix a block idempotent e of FN such that $be \neq 0$. Then $b = \sum_{g \in G/S} {}^{g}e$, where $S := \mathrm{stab}_G(e)$, and the idempotent e is also a block idempotent of FS. Let Q be a Sylow p-subgroup of S. Then Q is a defect group of the block idempotents e of FS, b of FG, and $\tilde{b}$ of $\mathbb{F}_pG$. Finally, set $\tilde{e} := \mathrm{Tr}_{\Gamma}(e)$, the block idempotent of $\mathbb{F}_pN$ determined by e and set $\tilde{S} := \mathrm{stab}_G(\tilde{e})$. Then $S \leqslant \tilde{S}$ and $\tilde{b} = \sum_{G/\tilde{S}} {}^{g}\tilde{e}$.

The group $\Gamma \times G$ acts on the block idempotents of FN. Set $X := \mathrm{stab}_{\Gamma \times G}(e)$. Since $\mathrm{stab}_G(e) = S$ we have $k_2(X) := \{g \in G \mid (1, g) \in X\} = S$. Similarly, $k_1(X) := \{\sigma \in \Gamma \mid (\sigma, 1) \in X\} = \mathrm{stab}_{\Gamma}(e)$. Next we determine the images of X under the projection maps $p_1 \colon \Gamma \times G \to \Gamma$ and $p_2 \colon \Gamma \times G \to G$.

5.1 Lemma *One has $p_2(X) = \tilde{S}$ and $S \trianglelefteq \tilde{S}$.*

Proof Let $g \in p_2(X)$. There exists $\sigma \in \Gamma$ such that ${}^{(\sigma,g)}e = e$. Therefore we have

$$\tilde{e} = \mathrm{Tr}_\Gamma(e) = \mathrm{Tr}_\Gamma({}^{(\sigma,g)}e) = \mathrm{Tr}_\Gamma({}^{\sigma}({}^{g}e)) = \mathrm{Tr}_\Gamma({}^{g}e) = {}^{g}\mathrm{Tr}_\Gamma(e) = {}^{g}\tilde{e}\,.$$

This shows that $g \in \mathrm{stab}_G(\tilde{e}) = \tilde{S}$ and hence that $p_2(X) \leqslant \tilde{S}$.

Now let $\tilde{s} \in \tilde{S}$. Then

$$\mathrm{Tr}_\Gamma({}^{\tilde{s}}e) = {}^{\tilde{s}}(\mathrm{Tr}_\Gamma(e)) = {}^{\tilde{s}}\tilde{e} = \tilde{e}\,.$$

Since the blocks e and ${}^{\tilde{s}}e$ have the same Galois trace, they must be Γ-conjugate, and therefore $\tilde{s} \in p_2(X)$. This proves the first statement. The second statement holds, since $k_2(X)$ is normal in $p_2(X)$ in general, see [Bc10, p. 24]. □

Next, set $e' := \sum_{\tilde{s}\in\tilde{S}/S} {}^{\tilde{s}}e$. Then e' is a block idempotent of $F\tilde{S}$ and $b = \sum_{g\in G/\tilde{S}} {}^{g}e'$.

5.2 Lemma *One has* $\mathrm{stab}_\Gamma(e') = \mathrm{stab}_\Gamma(b) = p_1(X)$. *Moreover,* $\tilde{S}/S \cong \mathrm{stab}_\Gamma(b)/\mathrm{stab}_\Gamma(e)$ *is cyclic.*

Proof We have $\mathrm{stab}_\Gamma(e') \leqslant \mathrm{stab}_\Gamma(b)$ since $b = \sum_{g\in G/\tilde{S}} {}^{g}e'$. Next, let $\sigma \in p_1(X)$. Then there exists $\tilde{s}_0 \in \tilde{S}$ such that $(\sigma, \tilde{s}_0) \in X$, and

$$e' = \sum_{\tilde{s}\in\tilde{S}/S} {}^{\tilde{s}}e = \sum_{\tilde{s}\in\tilde{S}/S} {}^{\tilde{s}}({}^{(\sigma,\tilde{s}_0)}e) = {}^{\sigma}\Big(\sum_{\tilde{s}\in\tilde{S}/S} {}^{\tilde{s}}({}^{\tilde{s}_0}e)\Big) = {}^{\sigma}e'\,,$$

where the last equation holds, because $S \trianglelefteq \tilde{S}$. This shows that $\sigma \in \mathrm{stab}_\Gamma(e')$ and hence $p_1(X) \leqslant \mathrm{stab}_\Gamma(e')$. Finally, let $\sigma \in \mathrm{stab}_\Gamma(b)$. Then ${}^{\sigma}b = b$ implies that ${}^{\sigma}(\sum_{g\in G/S} {}^{g}e) = \sum_{g\in G/S} {}^{g}e$. Therefore there exists $g \in G$ such that ${}^{(\sigma,g)}e = e$, i.e., $\sigma \in p_1(X)$. The proof of the first statement is now complete. The second statement follows from the general isomorphism $p_1(X)/k_1(X) \cong p_2(X)/k_2(X)$, see [Bc10, p. 24], and since Γ is cyclic. □

5.3 Lemma *One has* $\mathrm{Tr}_\Gamma(e') = \tilde{e}$. *In particular,* $\tilde{e}$ *is a block idempotent of* $\mathbb{F}_p\tilde{S}$.

Proof By Lemma 5.2 and since $k_1(X) = \mathrm{stab}_\Gamma(e)$, we have

$$\begin{aligned}\mathrm{Tr}_\Gamma(e') &= \sum_{\sigma\in\Gamma/p_1(X)} {}^{\sigma}e' = \sum_{\sigma\in\Gamma/p_1(X)} \sum_{\tilde{s}\in\tilde{S}/S} {}^{\sigma}({}^{\tilde{s}}e) = \sum_{\sigma\in\Gamma/p_1(X)} \sum_{\tau\in p_1(X)/k_1(X)} {}^{\sigma}({}^{\tau}e)\\ &= \sum_{\sigma\in\Gamma/k_1(X)} {}^{\sigma}e = \mathrm{Tr}_\Gamma(e) = \tilde{e}\,,\end{aligned}$$

as desired. The third equation holds, since the classes of $\tilde{s}$ and τ correspond under the isomorphism $p_2(X)/k_2(X) \cong p_1(X)/k_1(X)$ if and only if $(\tau, \tilde{s}) \in X$, see [Bc10, p. 24]. □

Let V denote the unique (up to isomorphism) simple FNe-module. By Theorem 2.6 and since $\mathrm{stab}_\Gamma(V) = \mathrm{stab}_\Gamma(e) = k_1(X)$, there exists a unique simple $\mathbb{F}_pN$-module $\widetilde{V}$ such that

$$\bigoplus_{\sigma\in\Gamma/k_1(X)} {}^{\sigma}V \cong F \otimes_{\mathbb{F}_p} \widetilde{V}\,. \tag{4}$$

Since $\tilde{e}$ acts as identity on the above direct sum, $\widetilde{V}$ is a simple $\mathbb{F}_pN\tilde{e}$-module. Since V is absolutely irreducible, it extends to a (unique up to isomorphism) simple FSe-module which we denote again by V. Similarly, each ${}^{\sigma}V$ can be viewed as FS-module, so that the left hand side in (4) has an $FS\tilde{e}$-module structure and is Γ-invariant. Again, by Theorem 2.6, the left hand side in (4) regarded as FS-module has an $\mathbb{F}_p$-form $W \in {}_{\mathbb{F}_pS\tilde{e}}\mathsf{mod}$. Restriction from S to N and the Deuring-Noether Theorem then imply that $\mathrm{Res}^S_N(W) \cong \widetilde{V}$. Thus, $\widetilde{V}$ extends to a simple $FS\tilde{e}$-module and (4) is an isomorphism of $FS\tilde{e}$-modules.

5.4 Proposition *The* $\mathbb{F}_p S\tilde{e}$*-module* $\widetilde{V}$ *extends to an* $\mathbb{F}_p\tilde{S}\tilde{e}$*-module.*

Proof By Fong's first reduction theorem, $W := \mathrm{Ind}_S^{\tilde{S}} V$ is the unique simple $F\tilde{S}e'$-module (up to isomorphism) and $\mathrm{stab}_\Gamma(W) = \mathrm{stab}_\Gamma(e') = p_1(X)$. By Theorem 2.6, there exists a simple $\mathbb{F}_p\tilde{S}$-module $\widetilde{W}$ such that $\bigoplus_{\sigma\in\Gamma/p_1(X)} {}^{\sigma}W \cong F\otimes_{\mathbb{F}_p}\widetilde{W}$. Restriction to S implies

$$\begin{aligned}\mathrm{Res}_S^{\tilde{S}}(F\otimes_{\mathbb{F}_p}\widetilde{W}) &\cong \mathrm{Res}_S^{\tilde{S}}\Big(\bigoplus_{\sigma\in\Gamma/p_1(X)} {}^{\sigma}W\Big) \cong \bigoplus_{\sigma\in\Gamma/p_1(X)} {}^{\sigma}(\mathrm{Res}_S^{\tilde{S}}W) \cong \bigoplus_{\sigma\in\Gamma/p_1(X)} {}^{\sigma}(\mathrm{Res}_S^{\tilde{S}}\mathrm{Ind}_S^{\tilde{S}}V)\\ &\cong \bigoplus_{\sigma\in\Gamma/p_1(X)} {}^{\sigma}\Big(\bigoplus_{\tilde{s}\in\tilde{S}/S} {}^{\tilde{s}}V\Big) \cong \bigoplus_{\sigma\in\Gamma/k_1(X)} {}^{\sigma}V \cong F\otimes_{\mathbb{F}_p}\widetilde{V}\,,\end{aligned}$$

since $\bigoplus_{\tilde{s}\in\tilde{S}/S} {}^{\tilde{s}}V \cong \bigoplus_{\tau\in p_1(X)/k_1(X)} {}^{\tau}V$, which follows from the argument at the end of the proof of the previous proposition. This shows that $\mathrm{Res}_S^{\tilde{S}}\widetilde{W} = \widetilde{V}$ and the result follows. ∎

5.5 Remark Proposition 5.4 extends the results of Michler [M73, Theorem 3.7] (z=1 in part(e)).

Now set $H := N_G(Q)$, which is again a p-nilpotent group, and set $M := O_{p'}(H)$, the largest normal p'-subgroup of H. Then

$$M = H\cap N = C_N(Q)\,.$$

Let c denote the block idempotent of FH which is in Brauer correspondence with b. Then, by Lemmas 4.2 and 5.2, $\mathrm{stab}_\Gamma(c) = \mathrm{stab}_\Gamma(b) = p_1(X)$ and $\tilde{c} := \mathrm{Tr}_\Gamma(c)$ is the Brauer correspondent of $\tilde{b}$.

Further, let f denote the block idempotent of FM whose irreducible module is the Glaubermann correspondent of the Q-stable irreducible module $V \in {}_{FN}\mathsf{mod}$. Then f is $QM = QC_N(Q) = N_S(Q) = N_G(Q)\cap S = H\cap S =: T$-stable and hence it remains a block idempotent of FT. By [A76], the block idempotents e of FS and f of FT are Brauer correspondents.

5.6 Lemma *One has* $c = \mathrm{Tr}_T^H(f)$ *and* $\mathrm{stab}_H(f) = T$.

Proof Since the block idempotents b and c are Brauer correspondents, we have

$$\begin{aligned}c = \mathrm{Br}_Q(b) = \mathrm{Br}_Q(\mathrm{Tr}_S^G(e)) &= \mathrm{Br}_Q\Big(\sum_{x\in Q\backslash G/S} \mathrm{Tr}^Q_{Q\cap {}^xS}({}^xe)\Big)\\ = \sum_{x\in Q\backslash G/S} \mathrm{Br}_Q\big(\mathrm{Tr}^Q_{Q\cap {}^xS}({}^xe)\big) &= \sum_{\substack{x\in Q\backslash G\\ Q\leqslant {}^xS}} \mathrm{Br}_Q({}^xe)\,.\end{aligned}$$

The condition $Q\leqslant {}^xS$ implies that ${}^{x^{-1}}Q\leqslant S$ and hence ${}^{x^{-1}}Q = {}^sQ$ for some $s\in S$ since Q is a Sylow p-subgroup of S. This means that $xs\in N_G(Q)$ and so $x\in N_G(Q)S$. Therefore the above sum can be written as

$$c = \sum_{x\in N_G(Q)/(N_G(Q)\cap S)} \mathrm{Br}_Q({}^xe) = \sum_{x\in H/T} {}^x\mathrm{Br}_Q(e) = \mathrm{Tr}_T^H(f)\,,$$

since $f = \mathrm{Br}_Q(e)$. This proves the first assertion. The group $\mathrm{stab}_H(f)$ has the group Q as a Sylow p-subgroup, since Q is a defect group of the bock c. This shows that $\mathrm{stab}_H(f) = QM = T$, as desired. ∎

Let $\tilde{f} := \mathrm{Tr}_\Gamma(f)$, $\tilde{T} := \mathrm{stab}_H(\tilde{f})$, $f' := \mathrm{Tr}_T^{\tilde{T}}(f)$ and $Y := \mathrm{stab}_{\Gamma\times H}(f)$. Since the blocks e and f are Brauer correspondents, we have

$$k_1(Y) = \mathrm{stab}_\Gamma(f) = \mathrm{stab}_\Gamma(e) = k_1(X)\,. \tag{5}$$

Moreover, by Lemmas 5.1 and 5.2,

$$\mathrm{stab}_\Gamma(f') = p_1(Y) = \mathrm{stab}_\Gamma(c) = \mathrm{stab}_\Gamma(b) = p_1(X) = \mathrm{stab}_\Gamma(e')\,, \tag{6}$$
$$p_2(Y) = \tilde{T} \quad \text{and} \quad k_2(Y) = T\,,$$

and therefore

$$\tilde{T}/T = p_2(Y)/k_2(Y) \cong p_1(Y)/k_1(Y) = p_1(X)/k_1(X) \cong p_2(X)/k_2(X) = \tilde{S}/S \tag{7}$$

which implies that $\tilde{T} = H \cap \tilde{S}$.

5.7 We recall Rickard's construction of a *splendid Rickard equivalence* between FSe and FQ, i.e., a bounded chain complex X of relatively $\Delta(Q)$-projective p-permutation (FSe, FQ)-bimodules such that $X \otimes_{FQ} X^\circ \cong FSe$ and $X^\circ \otimes_{FS} X \cong FQ$ in the homotopy categories of (FSe, FSe)-bimodules and (FQ, FQ)-bimodules, respectively, where FSe and FQ are considered as chain complexes concentrated in degree 0. For more details we refer the reader to [R96].

Set $\Delta_Q S := \{(nq, q) : n \in N, q \in Q\} \leqslant S \times Q$ and note that $p_1 \colon S \times Q \to S$ restricts to an isomorphism $\Delta_Q S \xrightarrow{\sim} S$. The module $\mathrm{Res}_Q^S V$ is a capped endopermutation FQ-module. In everything that follows, we suppose that

$$\mathrm{Res}_Q^S V \textit{ has an endosplit } p\textit{-permutation resolution } X_V. \tag{8}$$

By the proof in [R96, Lemma 7.7], see also Remark (a) at the end of Section 7 in [R96], the induced complex $\mathrm{Ind}_Q^S(X_V)$ is an endosplit p-permutation resolution of $\mathrm{Ind}_Q^S \mathrm{Res}_Q^S V$ as FS-modules. Since $V \mid \mathrm{Ind}_Q^S \mathrm{Res}_Q^S V$, there exists a direct summand Y_V of $\mathrm{Ind}_Q^S X_V$ such that Y_V is an endosplit p-permutation resolution of V as an FS-module, and we may choose Y_V to be contractible-free, see Remark 3.2(c) and (b). The induced chain complex $\mathrm{Ind}_{\Delta_Q S}^{S\times Q} Y_V$ is then a splendid Rickard equivalence between FSe and FQ, see [R96, Theorem 7.8] and its subsequent Remark (a).

5.8 Proposition *Suppose that* $\mathrm{stab}_\Gamma(e) = \mathrm{stab}_\Gamma(b)$ *and that* $\mathrm{Res}_Q^S V$ *has an endosplit* p*-permutation resolution.*

(a) *There exists a splendid Rickard equivalence between* $\mathbb{F}_p S\tilde{e}$ *and* $\mathbb{F}_p T\tilde{f}$.

(b) *There exists a splendid Rickard equivalence between* $\mathbb{F}_p G\tilde{b}$ *and* $\mathbb{F}_p H\tilde{c}$.

Proof (a) The equality $\mathrm{stab}_\Gamma(e) = \mathrm{stab}_\Gamma(b)$ implies that we have

$$\mathrm{stab}_\Gamma(f) = \mathrm{stab}_\Gamma(c) = \mathrm{stab}_\Gamma(b) = \mathrm{stab}_\Gamma(e)\,, \quad \tilde{S} = S\,, \quad \tilde{T} = T\,, \quad e' = e \quad \text{and} \quad f' = f\,,$$

by (5) and (6). Let $\mathbb{F}_p[e]$ denote the smallest field containing the coefficients of the idempotent e. Then $F' := \mathbb{F}_p[e] = \mathbb{F}_p[f] \subseteq F$. By Corollary 2.5, there exists an absolutely simple $F'N$-module V' such that $V \cong F \otimes_{F'} V'$, the unique simple module in the block FNe. Since e is S-stable, also V' extends to an $F'S$-module that we again denote by V'. Then $V \cong F \otimes_{F'} V'$ also as FSe-modules. Since $V \in {}_{FS}\mathsf{mod}$ has an endosplit p-permutation resolution, also $V' \in {}_{F'S}\mathsf{mod}$ has an endosplit p-permutation resolution $X' \in Ch({}_{F'S}\mathsf{mod})$, see Remark 3.2(d). Since F' is a splitting field of V' as $F'N$-module, we may use the results from Theorem 7.8 and its subsequent Remark (a) in [R96] in order to see that $\mathrm{Ind}_{\Delta_Q S}^{S\times Q}(X')$ is a splendid Rickard equivalence between $F'Se$ and $F'Q$. Using the unique F'-form $U' \in {}_{F'M}\mathsf{mod}$ of $U \in {}_{FM}\mathsf{mod}$ and its unique extension to an $F'T$-mdoule, we similarly obtain that $\mathrm{Ind}_{\Delta_Q T}^{T\times Q}(U')$ induces a splendid Morita equvivalence between $F'Tf$ and $F'Q$. Thus, the chain complex

$$Z' = \mathrm{Ind}_{\Delta_Q S}^{S\times Q}(X') \otimes_{FQ} \mathrm{Ind}_{(\Delta_Q T)^\circ}^{Q\times T}(U')^\circ$$

is a splendid Rickard equivalence between $F'Se$ and $F'Tf$. The result now follows from [KL18, Theorem 6.5].

(b) The p-permutation bimodule $\mathbb{F}_p G\tilde{e}$ induces a Morita equivalence, hence a splendid Rickard equivalence, between $\mathbb{F}_p G\tilde{b}$ and $\mathbb{F}_p S\tilde{e}$. Similarly, the bimodule $\mathbb{F}_p H\tilde{f}$ induces a splendid Rickard equivalence between $\mathbb{F}_p H\tilde{c}$ and $\mathbb{F}_p T\tilde{f}$. The result follows now from Part (a). □

5.9 Remark By the classification of indecomposable capped endopermutation modules, the hypothesis in (8) is satisfied if p is odd, or if $p = 2$ and Q does not have a subquotient isomorphic to the quaternion group of order 8. See [T07] for more details. Therefore, Theorem A follows from Proposition 5.8.

5.10 (a) By Fong's first reduction theorem, the $(F\tilde{S}e', FSe)$-bimodule $F\tilde{S}e$ induces a Morita equivalence between $F\tilde{S}e'$ and FSe. Hence the complex $F\tilde{S}e \otimes_{FS} \mathrm{Ind}_{\Delta_Q S}^{S\times Q} Y_V$ gives a splendid Rickard equivalence between $F\tilde{S}e'$ and FQ.

(b) For any FS-module M, let $M \otimes_F FQ$ be the (FS, FQ)-bimodule, given by

$$s(m \otimes x)y := sm \otimes qxy, \quad \text{for } s = nq \in S,\ n \in N,\ m \in M, \text{ and } x, y, q \in Q.$$

It is straightforward to check that the map

$$\phi_M : M \otimes_F FQ \to F[S \times Q] \otimes_{F\Delta_Q S} M$$
$$v \otimes q \mapsto (1, q^{-1}) \otimes v$$

is an isomorphism of (FS, FQ)-bimodules and that it is natural in M. Therefore, it yields an isomorphism

$$Y_V \otimes_F FQ \cong \mathrm{Ind}_{\Delta_Q S}^{S\times Q} Y_V$$

of chain complexes of (FSe, FQ)-bimodules.

(c) Let U be the simple FM-module belonging to the block idempotent f. Since Q is normal in H, we have $T = Q \times M$. Thus, the unique extension of U to T (with Q acting trivially on U) is a p-permutation FT-module and plays the same role as the complex Y_V. Similar as in (a), the bimodule $F\tilde{T}f \otimes_{FT} \mathrm{Ind}_{\Delta_Q T}^{T\times Q} U$ induces a splendid Rickard equivalence between $F\tilde{T}f'$ and FQ.

(d) Altogether, the complex

$$Z := F\tilde{S}e \otimes_{FS} \mathrm{Ind}_{\Delta_Q S}^{S\times Q} Y_V \otimes_{FQ} \mathrm{Ind}_{(\Delta_Q T)^\circ}^{Q\times T}(U)^\circ \otimes_{FT} fF\tilde{T}$$

induces a splendid Rickard equivalence between $F\tilde{S}e'$ and $F\tilde{T}f'$. Here, $(\Delta_Q T)^\circ := \{(q,t) \in Q \times T \mid (t,q) \in \Delta_Q T\}$. Set

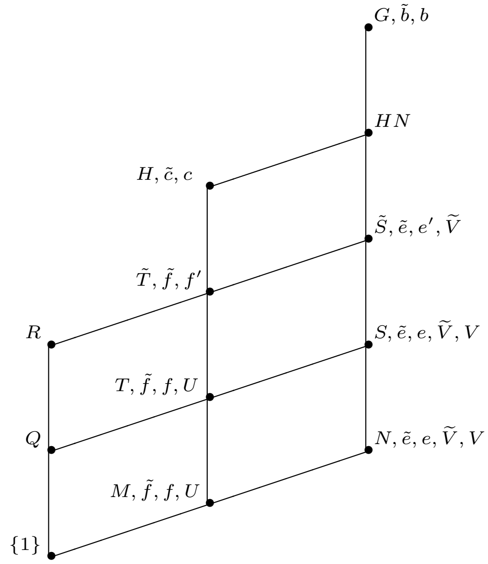
$$\omega := \sum_{n\in\mathbb{Z}} (-1)^n [Z_n] \in T^\Delta(F\tilde{S}e', F\tilde{T}f')\,.$$

By [BX08, Theorem 1.5], ω is a p-permutation equivalence between $F\tilde{S}e'$ and $F\tilde{T}f'$. Moreover, the isomorphism in (b) implies that

$$\begin{aligned} Z &= F\tilde{S}e \otimes_{FS} \mathrm{Ind}_{\Delta_Q S}^{S\times Q} Y_V \otimes_{FQ} \mathrm{Ind}_{(\Delta_Q T)^\circ}^{Q\times T}(U)^\circ \otimes_{FT} fF\tilde{T} \\ &\cong F\tilde{S}e \otimes_{FS} (Y_V \otimes_F FQ) \otimes_{FQ} (FQ \otimes_F U^\circ) \otimes_{FT} fF\tilde{T} \\ &\cong F\tilde{S}e \otimes_{FS} (Y_V \otimes_F FQ \otimes_F U^\circ) \otimes_{FT} fF\tilde{T}\,. \end{aligned}$$

Let R be a Sylow p-subgroup of $\tilde{T}$ containing Q. Then $\tilde{T} = RM$, and, by (7), R is also a Sylow p-subgroup of $\tilde{S}$ so that $\tilde{S} = RN$.

The following diagram depicts the subgroups, block idempotents, and modules introduced so far.



5.11 Lemma *For every $r \in R$, one has an isomorphism*

$$F\tilde{S}e \otimes_{FS} (Y_V \otimes_F FQ \otimes_F U^\circ) \otimes_{FT} fF\tilde{T} \cong F\tilde{S}\,{}^re \otimes_{FS} ({}^rY_V \otimes_F FQ \otimes_F {}^r(U^\circ)) \otimes_{FT} {}^rfF\tilde{T}$$

of chain complexes of $(F\tilde{S}e', F\tilde{T}f')$-bimodules.

Proof For any $M \in {}_{FS}\mathsf{mod}$, consider the map

$$F\tilde{S}e \otimes_{FS} (M \otimes_F FQ \otimes_F U^\circ) \otimes_{FT} fF\tilde{T} \to F\tilde{S}\,{}^re \otimes_{FS} ({}^rM \otimes_F FQ \otimes_F {}^r(U^\circ)) \otimes_{FT} {}^rfF\tilde{T}\,,$$

mapping $a \otimes (y \otimes q \otimes u) \otimes b$ to $ar^{-1} \otimes (y \otimes rqr^{-1} \otimes u) \otimes rb$. It is straightforward to check that it is well-defined, an isomorphism of $(F\tilde{S}e', F\tilde{T}f')$-bimodules, and functorial in M. Thus, it yields the desired isomorphism of chain complexes. □

5.12 For the rest of the paper we assume that there exists $W \in {}_{\mathbb{F}_pQ}\mathsf{mod}$ such that

$$\mathrm{Res}^S_Q V \cong F \otimes_{\mathbb{F}_p} W \text{ and that } W \text{ has an endosplit } p\text{-permutation resolution } X_W. \qquad (9)$$

Then the chain complex $F \otimes_{\mathbb{F}_p} X_W$ is an endosplit p-permutation resolution of $\mathrm{Res}^S_Q V$ and we assume from now on that $X_V = F \otimes_{\mathbb{F}_p} X_W$.

Note that that if Q is abelian then (9) is satisfied. In fact, every indecomposable endopermutation module for an abelian p-group is a direct summand of tensor products of inflations of Heller translates of the trivial module of quotient groups (see [D78] or [T07]), and every indecomposable endopermutation module (over any base field) is absolutely indecomposable (see Theorem 6.6 in the first paper [D78]). It follows that $\mathrm{Res}^S_Q(V)$ has an $\mathbb{F}_p$-form $W \in {}_{\mathbb{F}_pQ}\mathsf{mod}$. Moreover, W has an endosplit p-permutation resolution X_W (see [R96, Theorem 7.2] whose proof is still valid over $\mathbb{F}_p$).

5.13 Proposition *Suppose that R is abelian. For any $\tilde{s} \in \tilde{S}$, one has an isomorphism ${}^{\tilde{s}}(\mathrm{Ind}_Q^S X_W) \cong \mathrm{Ind}_Q^S X_W$ of complexes of $\mathbb{F}_pS$-modules. In particular, for any $\tilde{s} \in \tilde{S}$, one has ${}^{\tilde{s}}(\mathrm{Ind}_Q^S \mathrm{Res}_Q^S V) \cong \mathrm{Ind}_Q^S \mathrm{Res}_Q^S V$ as FS-modules.*

Proof The complex $\mathrm{Ind}_Q^S X_W$ is isomorphic to a complex whose terms are direct sums of permutation $\mathbb{F}_pS$-modules of the form $\mathbb{F}_p[S/Q_0]$ where $Q_0 \leqslant Q$ and whose differentials are $\mathbb{F}_p$-linear combination of maps of the form $f_t \colon \mathbb{F}_p[S/Q_1] \to \mathbb{F}_p[S/Q_2]$, $Q_1 \mapsto \sum_{sQ_2 \in Q_1 t Q_2} sQ_2$, for some $t \in Q$, with $Q_1, Q_2 \leqslant Q$. Let $\tilde{s} \in \tilde{S} = RN$ and write $\tilde{s} = rn$ for some $r \in R$ and $n \in N$. For any $Q_0 \leqslant Q$, one has an isomorphism $\mathbb{F}_p[S/Q_0] \to {}^{\tilde{s}}\mathbb{F}_p[S/Q_0]$ of $\mathbb{F}_pS$-modules given by $sQ_0 \mapsto sn^{-1}Q_0$. Moreover, a quick computation shows that this isomorphism commutes with the above maps f_t, since R is abelian. Therefore we have ${}^{\tilde{s}}(\mathrm{Ind}_Q^S X_W) \cong \mathrm{Ind}_Q^S X_W$. For the last assertion note that this also implies that ${}^{\tilde{s}}(F \otimes_{\mathbb{F}_p} \mathrm{Ind}_Q^S X_W) \cong F \otimes_{\mathbb{F}_p} \mathrm{Ind}_Q^S X_W$. Since the module $\mathrm{Ind}_Q^S \mathrm{Res}_Q^S V$ is the homology of the complex $F \otimes_{\mathbb{F}_p} \mathrm{Ind}_Q^S X_W$ the result follows. ∎

5.14 Lemma *Suppose that R is abelian. Then one has $\mathrm{stab}_\Gamma(Z) = \mathrm{stab}_\Gamma(\omega) = p_1(X) = \mathrm{stab}_\Gamma(e') = \mathrm{stab}_\Gamma(f') = p_1(Y)$.*

Proof Note that $p_1(X) = \mathrm{stab}_\Gamma(e') = \mathrm{stab}_\Gamma(f') = p_1(Y)$ hold by (6).

Since the complex Z induces a splendid Rickard equivalence between $F\tilde{S}e'$ and $F\tilde{T}f'$, the inclusion $\mathrm{stab}_\Gamma(Z) \leqslant \mathrm{stab}_\Gamma(e')$ is immediate. Thus, $\mathrm{stab}_\Gamma(Z) \leqslant p_1(X)$. Conversely, if $\sigma \in p_1(X)$, then ${}^\sigma e = {}^{\tilde{s}}e$ for some $\tilde{s} \in \tilde{S}$. Write $\tilde{s} = rn$ for some $r \in R$ and $n \in N$ and note that we have ${}^\sigma e = {}^r e$. This implies that ${}^\sigma V \cong {}^r V$ as FS-modules. By Proposition 5.13, we have ${}^\sigma(F \otimes_{\mathbb{F}_p} \mathrm{Ind}_Q^S X_W) \cong F \otimes_{\mathbb{F}_p} \mathrm{Ind}_Q^S X_W \cong {}^r(F \otimes_{\mathbb{F}_p} \mathrm{Ind}_Q^S X_W)$ as complexes of FS-modules and ${}^\sigma(\mathrm{Ind}_Q^S \mathrm{Res}_Q^S V) \cong \mathrm{Ind}_Q^S \mathrm{Res}_Q^S V \cong {}^r(\mathrm{Ind}_Q^S \mathrm{Res}_Q^S V)$ as FS-modules. Therefore Lemma 3.3 implies that ${}^\sigma Y_V \cong {}^r Y_V$ as complexes of FS-modules, as Y_V was chosen to be contractible-free, see 5.7. Since the idempotents e of FS and f of FT are Brauer correspondents, also ${}^r e$ and ${}^r f$ are Brauer correspondents. Since the Galois action commutes with the Brauer correspondence, ${}^\sigma e = {}^r e$ implies ${}^\sigma f = {}^r f$. Therefore we have ${}^\sigma U \cong {}^r U$ as FT-modules. By Lemma 5.11, we obtain

$$\begin{aligned} {}^\sigma Z &\cong {}^\sigma\Big(F\tilde{S}e \otimes_{FS} (Y_V \otimes_F FQ \otimes_F U^\circ) \otimes_{FT} fF\tilde{T}\Big) \\ &\cong F\tilde{S}\,{}^\sigma e \otimes_{FS} ({}^\sigma Y_V \otimes_F FQ \otimes_F {}^\sigma U^\circ) \otimes_{FT} {}^\sigma f F\tilde{T} \\ &\cong F\tilde{S}\,{}^r e \otimes_{FS} ({}^r Y_V \otimes_F FQ \otimes_F {}^r U^\circ) \otimes_{FT} {}^r f F\tilde{T} \\ &\cong F\tilde{S}e \otimes_{FS} (Y_V \otimes_F FQ \otimes_F U^\circ) \otimes_{FT} fF\tilde{T} \cong Z\,. \end{aligned}$$

This proves that $\mathrm{stab}_\Gamma(Z) = p_1(X)$.

Since ω is a p-permutation equivalence between $F\tilde{S}e'$ and $F\tilde{T}f'$, the inclusion $\mathrm{stab}_\Gamma(\omega) \leqslant \mathrm{stab}_\Gamma(e') = p_1(X)$ is clear. The inclusion $\mathrm{stab}_\Gamma(Z) \leqslant \mathrm{stab}_\Gamma(\omega)$ is immediate, and the proof is complete. ∎

5.15 Corollary *Suppose that R is abelian.*

(a) *There exists a p-permutation equivalence between $\mathbb{F}_p\tilde{S}\tilde{e}$ and $\mathbb{F}_p\tilde{T}\tilde{f}$.*

(b) *There exists a p-permutation equivalence between $\mathbb{F}_pG\tilde{b}$ and $\mathbb{F}_pH\tilde{c}$.*

Proof (a) By Lemma 5.14 we have $\mathrm{stab}_\Gamma(\omega) = \mathrm{stab}_\Gamma(e') = \mathrm{stab}_\Gamma(f')$. Hence by Lemma 4.3 there exists a p-permutation equivalence between $\mathbb{F}_p\tilde{S}\tilde{e}$ and $\mathbb{F}_p\tilde{T}\tilde{f}$.

(b) The p-permutation bimodule $\mathbb{F}_p G\tilde{e}$ induces a Morita equivalence, hence a p-permutation equivalence, between $\mathbb{F}_p G\tilde{b}$ and $\mathbb{F}_p\tilde{S}\tilde{e}$. Similarly, the bimodule $\mathbb{F}_p H\tilde{f}$ induces a p-permutation equivalence between $\mathbb{F}_p H\tilde{c}$ and $\mathbb{F}_p\tilde{T}\tilde{f}$. The result follows now from Part (a). □

5.16 Remark If one had a descent result for splendid Rickard equivalences analogous to Lemma 4.3, one would also obtain a splendid Rickard equivalence between $\mathbb{F}_p G\tilde{b}$ and $\mathbb{F}_p H\tilde{c}$ in the above corollary, because Lemma 5.14 includes $\mathrm{stab}_\Gamma(Z)$, while in the proof of the above corollary we only used the statement about $\mathrm{stab}_\Gamma(\omega)$. In order to prove such a descent result one would need a descent result for homomorphisms between p-permutation modules.

Moreover, the approach in the proof of Proposition 5.8 does not work, since the first Fong reduction only gives an equivalence between $\mathbb{F}_p G\tilde{b}$ and $\mathbb{F}_p\tilde{S}\tilde{e} = \mathbb{F}_p\tilde{S}\tilde{e'}$, and not between $\mathbb{F}_p G\tilde{b}$ and $\mathbb{F}_p S\tilde{e}$. In order to apply the descent result from [KL18], one would first need to descend the chain complex Z from 5.10(d) to $\mathbb{F}_p[e']$. But we could not modify the approach from the proof of Proposition 5.8 to descend to $\mathbb{F}_p[e']$, unless $e = e'$ which is equivalent to $Q = R$ and to $\mathbb{F}_p[b] = \mathbb{F}_p[e]$.

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