The A-fibered Burnside Ring as A-Fibered Biset Functor in Characteristic Zero*

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Abstract

Let A be an abelian group and let $\mathbb{K}$ be a field of characteristic zero containing roots of unity of all orders equal to finite element orders in A. In this paper we prove fundamental properties of the A-fibered Burnside ring functor $B^A_{\mathbb{K}}$ as an A-fibered biset functor over $\mathbb{K}$. This includes a description of the composition factors of $B^A_{\mathbb{K}}$ and the lattice of subfunctors of $B^A_{\mathbb{K}}$ in terms of what we call B^A-pairs and a poset structure on their isomorphism classes. Unfortunately, we are not able to classify B^A-pairs. The results of the paper extend results of Coşkun and Yılmaz for the A-fibered Burnside ring functor restricted to p-groups and results of Bouc in the case that A is trivial, i.e., the case of the Burnside ring functor as a biset functor over fields of characteristic zero. In the latter case, B^A-pairs become Bouc's B-groups which are also not known in general.

1 Introduction

Let A be a finite group and let k be a commutative ring. An A-fibered biset functor F over k is, informally speaking, a functor that assigns to each finite group G a k-module $F(G)$ together with maps $\mathrm{res}^G_H\colon F(G) \to F(H)$ and $\mathrm{ind}^G_H\colon F(H) \to F(G)$, whenever $H \leqslant G$, called restriction and induction, maps $\mathrm{inf}^G_{G/N}\colon F(G/N) \to F(G)$ and $\mathrm{def}^G_{G/N}\colon F(G) \to F(G/N)$, whenever N is a normal subgroup of G, called inflation and deflation, and maps $\mathrm{iso}_f\colon F(G) \to F(H)$, whenever $f\colon G \to H$ is an isomorphism. Moreover, the abelian group $G^* := \mathrm{Hom}(G, A)$ acts k-linearly on $F(G)$ for every finite group G. These operations satisfy natural relations. Standard examples are various representation rings of KG-modules, for a field K and $A = K^\times$. In this case, G^* is the group of one-dimensional KG-modules acting by multiplication on these representation rings. In [BC18] the simple A-fibered biset functors were parametrized. If A is the trivial group then one

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obtains the well-established theory of biset functors, see [Bc10] as special case. A-fibered biset functors over k can also be interpreted as the modules over the Green biset functor B_k^A, where $B_k^A(G)$ is the A-fibered Burnside ring of G over k (also called the K-monomial Burnside ring of G over k, when $A = K^\times$ for a field K). Another natural example of A-fibered biset functors (without deflation) is the unit group functor $G \mapsto B^A(G)^\times$. This structure was established in a recent paper by Bouc and Mutlu, see [BM19] and generalizes the biset functor structure on the unit group $B(G)^\times$ of the Burnside ring.

Representation rings carry more structure when viewed as A-fibered biset functors compared to the biset functor structure. One of the goals is to understand their composition factors as such functors. By various induction theorems (e.g. by Brauer, Artin, Conlon) some of these representation rings can be viewed as quotient functors of the functor B_k^A for appropriate A and k. Thus, it is natural to first investigate the lattice of subfunctors of B_k^A and its composition factors. This is the objective of this paper, where k is a field of characteristic zero containing sufficiently many roots of unity. Coşkun and Yılmaz achieved this already in [CY19] for the same functor category restricted to finite p-groups for fixed p, and for A being a cyclic p-group. The choice of A being cyclic allows them to use results on primitive idempotents and species of $B_k^A(G)$ from [Ba04], which were based on embedding A into $k^\times$, for k a field that is large enough. Our choice of A is more general (using only that G^* is finite for all finite groups G), thereby requiring a more complicated parametrizing set for the species and primitive idempotents of $B_k^A(G)$. This way the roles of A and k are kept as separate as possible while still implying that $B_k^A(G)$ is split semisimple.

The paper is arranged as follows. In Section 2 we recall basic facts about the A-fibered Burnside ring $B_k^A(G)$, about fibered biset functors, and the definition of B_k^A as fibered biset functor. In Section 3 we parametrize the set of primitive idempotents of $B_{\mathbb{K}}^A(G)$ over a field $\mathbb{K}$ of characteristic 0 which contains enough roots of unity in relation to the finite element orders of A. We also derive an explicit formula for these idempotents, using results from [BRV19] on the $-_+$ construction. We take advantage of the fact that the Green biset functor $B_{\mathbb{K}}^A$ arises as the $-_+$ construction of the Green biset functor $G \mapsto \mathbb{K}G^*$. Interestingly, the idempotent formula we derive in Theorem 3.2 is different from the one given by Barker in [Ba04] when specializing to the more restrictive cases of A considered there. It is used as a crucial tool in the following sections. In Section 4 we provide formulas for the action of inductions, restrictions, inflations, deflations, isomorphisms and twists by $\phi \in G^*$ on these idempotents. Crucial among those is the action of $\mathrm{def}^G_{G/N}$, which maps a primitive idempotent of $B_{\mathbb{K}}^A(G)$ to a scalar multiple of a primitive idempotent of $B_{\mathbb{K}}^A(G/N)$. After establishing three technical lemmas in Section 5, this mysterious scalar is studied in more depth in Section 6. The vanishing of this scalar is a condition that leads to the notion of a B^A-pair (G, Φ), where G is a finite group and $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$, in Section 7. There, we also study particular subfunctors $E_{(G,\Phi)}$ of $B_{\mathbb{K}}^A$. In Section 8, we show that every subfunctor of $B_{\mathbb{K}}^A$ is a sum of some of the functors $E_{(G,\Phi)}$ and that the subfunctors of $B_{\mathbb{K}}^A$ are in bijection with the set of subsets of isomorphism classes of B^A-pairs that are closed from above with respect to a natural partial order $\preccurlyeq$, see Theorem 8.8. In Section 9 we show that the composition factors of $B_{\mathbb{K}}^A$ are parametrized by isomorphism classes of B^A-pairs. For a given B^A-pair, we also determine the parameter (a quadruple) of the corresponding composition factor in terms of the parametrization of all simple A-fibered biset functors over $\mathbb{K}$ given in [BC18]. Finally, in Section 10, we consider the special case that A is a subgroup of $\mathbb{K}^\times$. In this case, a natural isomorphism $G/O(G) \xrightarrow{\sim} \mathrm{Hom}(G^*, \mathbb{K}^\times)$ for a normal subgroup $O(G)$ of G depending on A, allows to reinterpret the set of B^A-pairs and makes our results compatible with the language and setup in [Ba04] and [CY19].

The approach in this paper follows closely the blueprint in [Bc10, Section 5] for the case $A = \{1\}$.

However, the presence of the fiber group A requires additional ideas and techniques to achieve the analogous results. The main technical problem is that a transitive A-fibered biset with stabilizer pair (U,ϕ), does in general not factor through the group $q(U)\cong p_i(U)/k_i(U)$, $i=1,2$, since ϕ is in general non-trivial when restricted to $k_1(U)\times k_2(U)$.

1.1 Notation For a finite group G we denote by $\exp(G)$ the exponent of G. If X is a left G-set, we write $x =_G y$ if two elements x and y of X belong to the same G-orbit. For $x\in X$, we denote by G_x or $\mathrm{stab}_G(x)$ the stabilizer of x in G. By X^G we denote the set of G-fixed points in X and by $[G\backslash X]$ a set of representatives of the G-orbits of X. For subgroups H and K of G, we denote by $[H\backslash G/K]$ a set of representatives of the (H,K)-double cosets of G.

For an abelian group A, we denote by $\mathrm{tor}(A)$ its subgroup of elements of finite order, and, for a ring R, we denote by $R^\times$ its group of units.

2 Prerequisites on the A-fibered biset functor B_k^A

Throughout this paper, we fix a multiplicatively written abelian group A. For any finite group G we set $G^*:=\mathrm{Hom}(G,A)$, which we view as abelian group with point-wise multiplication. We will freely use the language of bisets and biset functors as developed in [Bc10, Chapters 2 and 3]. Throughout, G, H, and K denote finite groups, and k denotes a commutative ring.

2.1 (a) The *A-fibered Burnside ring* $B^A(G)$ of G is defined as the Grothendieck group of the category of A-fibered left G-sets, see [BC18, 1.1 and 1.7]. It has a standard $\mathbb{Z}$-basis consisting of the G-orbits $[U,\phi]_G$ of pairs (U,ϕ), where $U\leqslant G$ and $\phi\in U^*$. The set $\mathcal{M}(G)=\mathcal{M}^A(G)$ of such pairs has a natural G-poset structure, see [BC18, 1.2]. The multiplication in $B^A(G)$ is given by

$$[U,\phi]_G\cdot[V,\psi]_G=\sum_{g\in[H\backslash G/V]}\big[U\cap {}^gV,\phi|_{U\cap {}^gV}\cdot({}^g\psi)|_{U\cap {}^gV}\big]_G\,, \tag{1}$$

and $[G,1]_G$ is the identity. We set $B_k^A(G):=k\otimes_{\mathbb{Z}}B^A(G)$. If A is the trivial group, we recover the Burnside ring $B(G)$ of G.

(b) We further set $B^A(G,H):=B^A(G\times H)$. This group can also be considered as the Grothendieck group of A-fibered (G,H)-bisets, see [BC18, 1.1 and 1.7]. The standard basis elements of $B^A(G,H)$ will be denoted by $\left[\frac{G\times H}{U,\phi}\right]$, for $(U,\phi)\in\mathcal{M}(G\times H)$. The tensor product of A-fibered bisets induces a bilinear map

$$-\underset{H}{\cdot}-\colon B^A(G,H)\times B^A(H,K)\to B^A(G,K)\,, \tag{2}$$

which behaves associatively. Its evaluation on standard basis elements is given by the formula (see [BC18, Corollary 2.5])

$$\left[\frac{G\times H}{U,\phi}\right]\underset{H}{\cdot}\left[\frac{H\times K}{V,\psi}\right]=\sum_{\substack{t\in[p_2(U)\backslash H/p_1(V)]\\ \phi_2|_{H_t}={}^t\psi_1|_{H_t}}}\left[\frac{G\times K}{U*{}^{(t,1)}V,\phi*{}^{(t,1)}\psi}\right]\,, \tag{3}$$

where $H_t:=k_2(U)\cap {}^tk_1(V)$. See [BC18, 1.2] for the definition of $p_i(U)$ and $k_i(U)$, for $i=1,2$, and [BC18, 2.3] for the definition of $U*{}^{(t,1)}V$ and $\phi*{}^{(t,1)}\psi$. Recall from the latter that $\phi_i\in k_i(U)^*$,

$i = 1, 2$, are defined as the unique elements satisfying $\phi|_{k_1(U)\times k_2(U)} = \phi_1 \times \phi_2^{-1}$, with ϕ_2^{-1} denoting the inverse of ϕ_2 in the group $k_2(U)^*$.

(c) Recall from [CY19, 3.1] that there exists a ring homomorphism $\Delta\colon B^A(G) \to B^A(G,G)$, $[U,\phi]_G \mapsto \left[\frac{G\times G}{\Delta(U),\Delta(\phi)}\right]$, where $\Delta(U) := \{(u,u) \mid u \in U\}$ and $\Delta(\phi)(u,u) := \phi(u)$. Here, $B^A(G,G)$ is considered as ring via the multiplication $\cdot_G$ from (2).

2.2 (a) The bilinear maps in (2) are used as the definition of composition in the k-linear category $\mathcal{C}_k^A$, whose objects are the finite groups and whose morphism set from H to G is $B_k^A(G,H)$. An *A-fibered biset functor over k* is a k-linear functor $F\colon \mathcal{C}_k^A \to {}_k\mathsf{Mod}$. Together with natural transformations, these functors form an abelian category $\mathcal{F}_k^A$. If A is the trivial group, one recovers the biset category $\mathcal{C}_k$ and the category of biset functors $\mathcal{F}_k$ over k, see [Bc10, Sections 3.1 and 3.2].

(b) The association $G \mapsto B_k^A(G)$, together with the bilinear maps in (2) applied to $K = 1$ (and using the canonical identifications of $H \cong H \times \{1\}$ and $G \cong G \times \{1\}$) defines the A-fibered biset functor $B_k^A \in \mathcal{F}_k^A$, which is the main object of study in this paper.

(c) If $f\colon A' \to A$ is a homomorphism between abelian groups, one obtains induced maps $\mathrm{Hom}(G,A') \to \mathrm{Hom}(G,A)$, $\mathcal{M}^{A'}(G) \to \mathcal{M}^A(G)$, ring homomorphisms $B^{A'}(G) \to B^A(G)$, and k-linear functors $\mathcal{C}_k^{A'} \to \mathcal{C}_k^A$ and $\mathcal{F}_k^A \to \mathcal{F}_k^{A'}$. In particular, when A' is the trivial group, we obtain embeddings $B(G,H) \subseteq B^A(G,H)$, $[(G\times H)/U] \mapsto \left[\frac{G\times H}{U,1}\right]$, and $\mathcal{C}_k \subseteq \mathcal{C}_k^A$. This way, we can view the elementary bisets res^G_H, ind^G_H, $\mathrm{inf}^G_{G/N}$, $\mathrm{def}^G_{G/N}$, iso_f, for $H \leqslant G$, $N \trianglelefteq G$, $f\colon G \xrightarrow{\sim} G'$, as elements in $B^A(G,H)$, $B^A(H,G)$, $B^A(G,G/N)$, $B^A(G/N/G)$, $B^A(G',G)$, respectively. One can verify quickly with (3) for $K = \{1\}$, that their operations under the A-fibered biset functor structure of B_k^A in (b) on standard basis elements are given by

$$\mathrm{res}^G_H([U,\phi]_G) = \sum_{g\in[H\backslash G/U]} [H \cap {}^gU, ({}^g\phi)|_{H\cap {}^gU}]_H\,, \quad \mathrm{ind}^G_H([V,\psi]_H) = [V,\psi]_G\,,$$

$$\mathrm{inf}^G_{G/N}([U/N,\phi]_{G/N}) = [U, \phi\circ\nu]_G\,,$$

where $\nu\colon U \to U/N$ is the natural epimorphism,

$$\mathrm{def}^G_{G/N}([U,\phi]_G) = \begin{cases} [UN/N, \tilde{\phi}]_{G/N}, & \text{if } U\cap N \leqslant \ker(\phi),\\ 0 & \text{otherwise,}\end{cases}$$

where $\tilde{\phi}(uN) := \phi(u)$ for $u \in U$, and

$$\mathrm{iso}_f([U,\phi]_G) = [f(U), \phi\circ f^{-1}|_{f(V)}]_{G'}\,.$$

In particular, for $g \in G$ and $H \leqslant G$, and $(U,\phi) \in \mathcal{M}(H)$, we have

$${}^g[U,\phi]_H := \mathrm{iso}_{c_g}([U,\phi]_H) = [{}^gU, {}^g\phi]_{{}^gH}\,,$$

where $c_g\colon H \xrightarrow{\sim} gHg^{-1}$ is the conjugation map. For $\lambda \in G^*$ and $(U,\phi) \in \mathcal{M}(G)$, one additionally has

$$\mathrm{tw}_\lambda([U,\phi]_G) = [U, \lambda|_U\cdot\phi]_G\,,$$

the *twist* by λ, coming from the application of $\mathrm{tw}_\lambda := \Delta([G,\lambda]_G) = \left[\frac{G\times G}{\Delta(G),\Delta(\lambda)}\right] \in B^A(G,G)$. It is easily verified that res^G_H, $\mathrm{inf}^G_{G/N}$ and iso_f are ring homomorphisms, and that $B_k^A(G)$ is a kG^*-algebra via tw.

Using the above notation, one obtains a canonical decomposition of a standard basis element of $B^A(G,H)$ into elementary bisets and a standard basis element for smaller groups.

2.3 Theorem *([BC18, Proposition 2.8]) Let $(U,\phi) \in \mathcal{M}(G\times H)$ and set $P := p_1(U)$, $Q := p_2(U)$, $K := \ker(\phi_1)$, and $L := \ker(\phi_2)$. Then $K \trianglelefteq P$, $L \trianglelefteq Q$, $K\times L \trianglelefteq U$, and*

$$\left[\frac{G\times H}{U,\phi}\right] = \mathrm{ind}_P^G \cdot_P \mathrm{inf}_{P/K}^P \cdot_{P/K} \left[\frac{P/K\times Q/L}{U/(K\times L),\bar{\phi}}\right] \cdot_{Q/L} \mathrm{def}_{Q/L}^Q \cdot_Q \mathrm{res}_L^H\,,$$

where $\bar{\phi} \in (U/(K\times L))^*$ is induced by ϕ and $U/(K\times L)$ is viewed as subgroup of $P/K\times Q/L$ via the canonical isomorphism $(P\times Q)/(K\times L) \cong P/K\times Q/L$.

2.4 Remark Let $\mathcal{D}_k$ be the subcategory of $\mathcal{C}_k$ with the same objects as $\mathcal{C}_k$, but with morphism sets generated by all elementary bisets, excluding inductions. In other words, $\mathrm{Hom}_{\mathcal{D}_k}(H,G) \subseteq B_k(G,H)$ is the free k-module generated by all standard basis elements $[(G\times H)/U]$ with $p_1(U) = G$. Mapping G to the group algebra kG^* defines a Green biset functor F on $\mathcal{D}_k$ over k in the sense of Romero's reformulation [R11, Definición 3.2.7, Lema 4.2.3] of Bouc's original definition [Bc10, Definition 8.5.1], with restriction, inflation and isomorphisms defined as usual, viewing $\mathrm{Hom}(-,A)$ as contravariant functor, and deflation defined by

$$\mathrm{def}_{G/N}^G(\phi) := \begin{cases} \bar{\phi}, \text{if } \phi|_N = 1, \\ 0, \text{otherwise,} \end{cases}$$

whenever N is a normal subgroup of G and $\phi \in G^*$. Here, $\bar{\phi} \in (G/N)^*$ is induced by ϕ. In fact, it is straightforward to check that all the relations in [Bc10, 1.1.3] that do not involve inductions are satisfied. Thus, we are in the situation of [BRV19, Theorem 7.3(a),(b)] and obtain via the $-_+$-construction a Green biset functor F_+ on $(\mathcal{D}_k)_+ = \mathcal{C}_k$. The Green biset functor F_+ is isomorphic to the Green biset functor B_k^A on $\mathcal{C}_k$. This follows from [BRV19, Theorem 4.7(c)] and by comparing the explicit formulas for the elementary biset operations in 2.2(c) with the explicit formulas in [BRV19, Remark 4.8]. We will use this point of view repeatedly in Sections 3 and 4.

3 Primitive idempotents of $B_{\mathbb{K}}^A(G)$

Throughout this section we assume that G is a finite group such that $H^* = \mathrm{Hom}(H,A)$ is a finite abelian group for every $H \leqslant G$. This is equivalent to $\mathrm{tor}_{\exp(G)}(A)$ being finite. Moreover, we assume that $\mathbb{K}$ is a splitting field of characteristic zero for all H^*, $H\leqslant G$. Note that this holds if and only if $\mathbb{K}$ has a root of unity of order $\exp(\mathrm{tor}_{\exp(G)}(A))$. Also note that in this case S^* is finite and $\mathbb{K}$ is a splitting field for S^*, for each subquotient S of G.

We define $\mathcal{X}(G)$ as the set of all pairs (H,Φ) with $H\leqslant G$ and $\Phi \in \mathrm{Hom}(H^*,\mathbb{K}^\times)$ and note that G acts on $\mathcal{X}(G)$ by conjugation: ${}^g(H,\Phi) := ({}^gH, {}^g\Phi)$, with ${}^g\Phi(\phi) := \Phi({}^{g^{-1}}\phi)$, for $g\in G$, $(H,\Phi)\in\mathcal{X}(G)$, and $\phi \in H^*$. The assumptions on $\mathbb{K}$ imply that, for any $H\leqslant G$,

$$\mathbb{K}H^* \to \prod_{\Phi\in\mathrm{Hom}(H^*,\mathbb{K}^\times)} \mathbb{K}\,, \quad a \mapsto (s_\Phi^H(a))_\Phi\,, \tag{4}$$

is an isomorphism of $\mathbb{K}$-algebras. Here, we $\mathbb{K}$-linearly extended Φ to a $\mathbb{K}$-algebra homomorphism

$$s_\Phi^H \colon \mathbb{K}H^* \to \mathbb{K}\,.$$

The first orthogonality relation implies that, for $\Psi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$, the element

$$e_\Psi^H := \frac{1}{|H^*|} \sum_{\phi \in H^*} \Psi(\phi^{-1})\phi \in \mathbb{K}H^* \tag{5}$$

is the primitive idempotent of $\mathbb{K}H^*$ which is mapped under the isomorphism in (4) to the primitive idempotent $\epsilon_\Psi^H \in \prod_\Phi \mathbb{K}$ whose Φ-component is $\delta_{\Phi,\Psi}$.

For any $H \leqslant G$ we consider the map

$$\pi_H \colon B_\mathbb{K}^A(H) \to \mathbb{K}H^*\,, \quad [U,\phi]_H \mapsto \begin{cases} \phi & \text{if } U = H, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen by the multiplication formula in (1) that π_H is a $\mathbb{K}$-algebra homomorphism and we obtain for every $(H,\Phi) \in \mathcal{X}(G)$, a $\mathbb{K}$-algebra homomorphism

$$s_{(H,\Phi)}^G := s_\Phi^H \circ \pi_H \circ \mathrm{res}_H^G \colon B_\mathbb{K}^A(G) \to B_\mathbb{K}^A(H) \to \mathbb{K}H^* \to \mathbb{K}\,.$$

3.1 Theorem *The map*

$$B_\mathbb{K}^A(G) \to \left(\prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K} \right)^G\,, \quad x \mapsto \left(s_{(H,\Phi)}^G(x)\right)_{(H,\Phi)}\,, \tag{6}$$

is a $\mathbb{K}$-algebra isomorphism. Here, G acts on $\prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K}$ by permuting the coordinates according to the G-action on $\mathcal{X}(G)$. In particular, every $\mathbb{K}$-algebra homomorphism $B_\mathbb{K}^A(G) \to \mathbb{K}$ is of the form $s_{(H,\Phi)}^G$ for some $(H,\Phi) \in \mathcal{X}(G)$. For $(H,\Phi),(K,\Psi) \in \mathcal{X}(G)$ one has $s_{(H,\Phi)}^G = s_{(K,\Psi)}^G$ if and only if $(H,\Phi) =_G (K,\Psi)$.

Proof By Theorem [BRV19, Theorems 6.1 and 7.3(c)] and using Remark 2.4, the *mark morphism*

$$m_G \colon B_\mathbb{K}^A(G) \to \left(\prod_{H\leqslant G} \mathbb{K}H^* \right)^G\,, \quad x \mapsto \left(\pi_H(\mathrm{res}_H^G(x))\right)_{H\leqslant G}\,, \tag{7}$$

is a homomorphism of $\mathbb{K}$-algebras and by Theorem [BRV19, Corollary 6.4] it is an isomorphism, since $|G|$ is invertible in $\mathbb{K}$. Here, G acts on $\prod_{H\leqslant G} \mathbb{K}H^*$ by ${}^g((a_H)_{H\leqslant G}) := ({}^g a_{g^{-1}Hg})_{H\leqslant G}$. Using the $\mathbb{K}$-algebra isomorphisms from (4), we obtain a G-equivariant $\mathbb{K}$-algebra isomorphism

$$\prod_{H\leqslant G} \mathbb{K}H^* \to \prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K}\,.$$

Applying the functor of G-fixed points to this isomorphism and composing it with the isomorphism in (7), we obtain the isomorphism in (6). The remaining assertions follow immediately. □

Clearly, for each $(H,\Phi) \in \mathcal{X}(G)$, we obtain a primitive idempotent $\epsilon_{(H,\Phi)}^G$ of the right hand side of the isomorphism (6). More precisely, $\epsilon_{(H,\Phi)}^G$ has entries equal to 1 at indices labelled by the G-conjugates of (H,Φ) and entries equal to 0 everywhere else. We denote the idempotent of $B_\mathbb{K}^A(G)$ corresponding to $\epsilon_{(H,\Phi)}^G$ by $e_{(H,\Phi)}^G \in B_\mathbb{K}^A(G)$. If (H,Φ) runs through a set of representatives of the

G-orbits of $\mathcal{X}(G)$ then $e^G_{(H,\Phi)}$ runs through the set of primitive idempotents of $B^A_{\mathbb{K}}(G)$, without repetition. Thus, we have

$$s^G_{(H,\Phi)}(e^G_{(K,\Psi)}) = \begin{cases} 1, & \text{if } (H,\Phi) =_G (K,\Psi), \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad x \cdot e^G_{(H,\Phi)} = s^G_{(H,\Phi)}(x) e^G_{(H,\Phi)}\,, \tag{8}$$

for any $(H,\Phi), (K,\Psi) \in \mathcal{X}(G)$ and any $x \in B^A_{\mathbb{K}}(G)$.

The following theorem gives an explicit formula for $e^G_{(H,\Phi)}$. A different formula for particular choices of A was given by Barker in [Ba04, Theorem 5.2]. For any $H \leqslant G$ and $a = \sum_{\phi \in H^*} a_\phi \phi \in \mathbb{K}H^*$ we will use the notation $[H,a]_G := \sum_{\phi\in H^*} a_\phi [H,\phi]_G \in B^A_{\mathbb{K}}(G)$. Moreover, $N_G(H,\Phi)$ denotes the stabilizer of (H,Φ) under G-conjugation.

3.2 Theorem *For $(H,\Phi) \in \mathcal{X}(G)$ one has*

$$\begin{aligned} e^G_{(H,\Phi)} &= \frac{1}{|N_G(H,\Phi)|} \sum_{K \leqslant H} |K|\mu(K,H)[K, \mathrm{res}^H_K(e^H_\Phi)]_G && (9)\\ &= \frac{1}{|N_G(H,\Phi)|} \sum_{\substack{K \leqslant H \\ \Phi|_{K^\perp}=1}} |K|\mu(K,H)[K, \mathrm{res}^H_K(e^H_\Phi)]_G && (10)\\ &= \frac{1}{|N_G(H,\Phi)| \cdot |H^*|} \sum_{\substack{K \leqslant H \\ \Phi|_{K^\perp}=1}} \sum_{\phi \in H^*} |K|\mu(K,H)\Phi(\phi^{-1})[K, \phi|_K]_G \in B^A_{\mathbb{K}}(G)\,, && (11) \end{aligned}$$

where $K^\perp := \{\phi \in H^* \mid \phi|_K = 1\} \leqslant H^*$ and μ is the Möbius function on the poset of all subgroups of G.

Proof We use the inversion formula of the $\mathbb{K}$-algebra isomorphism (7) from [BRV19, Proposition 6.3] and obtain

$$e^G_{(H,\Phi)} = \frac{1}{|G|} \sum_{L \leqslant K \leqslant G} |L|\mu(L,K)[L, \mathrm{res}^K_L(a_K)]_G\,, \tag{12}$$

with $a_K \in \mathbb{K}K^*$, $K \leqslant G$, given by $a_H = \sum_{x \in [N_G(H)/N_G(H,\Phi)]} {}^x e^H_\Phi$, $a_{{}^gH} = {}^g a_H$, for any $g \in G$, and $a_K = 0$ for all K not G-conjugate to H. Thus, in the above sum, for K it suffices to consider only over subgroups that are G-conjugate to H. We obtain

$$e^G_{(H,\Phi)} = \frac{1}{|G|} \sum_{x \in [G/N_G(H)]} \sum_{L \leqslant {}^xH} |L|\mu(L, {}^xH)[L, \mathrm{res}^{{}^xH}_L({}^x a_H)]_G\,. \tag{13}$$

Replacing L with ${}^xK$, for $K \leqslant H$, we see that the sum over L is independent of x and we obtain

$$e^G_{(H,\Phi)} = \frac{[G : N_G(H)]}{|G|} \sum_{K \leqslant H} |K|\mu(K,H)[K, \mathrm{res}^H_K(a_H)]_G\,. \tag{14}$$

Substituting $a_H = \sum_{g \in [N_G(H)/N_G(H,\Phi)]} {}^g e^H_\Phi$ and using the same argument as above, we obtain the formula in (9). In order to prove the formula in (10) it suffices to show that $\mathrm{res}^H_K(e^H_\Phi) = 0$ if $\Phi|_{K\perp} \neq 1$. Substituting the formula (5) for e^H_Φ, we obtain

$$\mathrm{res}^H_K(e^H_\Phi) = \frac{1}{|H^*|} \sum_{\phi \in H^*} \Phi(\phi^{-1})\phi|_K\,. \tag{15}$$

Note that $K^\perp = \ker(\mathrm{res}^H_K\colon H^* \to K^*)$ and choose for every $\psi \in \mathrm{im}(\mathrm{res}^H_K) \leqslant K^*$ an element $\tilde\psi \in H^*$ with $\tilde\psi|_K = \psi$. Then the right hand side in (15) is equal to

$$\frac{1}{|H^*|} \sum_{\psi\in\mathrm{im}(\mathrm{res}^H_K)} \sum_{\lambda\in K^\perp} \Phi(\tilde\psi^{-1}\lambda^{-1})\psi = \frac{1}{|H^*|} \sum_{\psi\in\mathrm{im}(\mathrm{res}^H_K)} \Phi(\tilde\psi^{-1})\Big(\sum_{\lambda\in K^\perp} \Phi(\lambda^{-1})\Big)\psi\,,$$

and it suffices to show that $\sum_{\lambda\in K^\perp} \Phi|_{K^\perp}(\lambda^{-1}) = 0$. But

$$\sum_{\lambda\in K^\perp} \Phi|_{K^\perp}(\lambda^{-1}) = [K^\perp : K^\perp \cap \ker(\Phi)] \sum_{\bar\lambda\in K^\perp/(K^\perp\cap\ker(\Phi))} \overline{\Phi|_{K^\perp}}(\bar\lambda^{-1})\,, \tag{16}$$

with an injective homomorphism from $\overline{\Phi|_{K^\perp}}\colon K^\perp/(K^\perp\cap\ker(\Phi)) \to \mathbb{K}^\times$. It follows that $K^\perp/(K^\perp\cap\ker(\Phi))$ is cyclic, say of order n. Our assumption on $\mathbb{K}$ implies that $\mathbb{K}$ has a primitive n-th root of unity. Moreover, since $\Phi|_{K^\perp} \neq 1$, we have $n > 1$. Thus the sum on the right hand side of (16) is equal to the sum of all n-th roots of unity in $\mathbb{K}$, which is 0. This proves Equation (10). Formula (11) is now immediate after substituting the formula for e^H_Φ. ∎

3.3 Remark If A' is the trivial subgroup of A we have $B^{A'}_{\mathbb{K}}(G) = B_{\mathbb{K}}(G)$, the Burnside algebra over $\mathbb{K}$. Using the functoriality properties in 2.2(c), we obtain a commutative diagram

$$\begin{array}{ccccc}
B_{\mathbb{K}}(G) & \xrightarrow{m_G} & \Big(\prod_{H\leqslant G} \mathbb{K}\Big)^G & \xrightarrow{\mathrm{id}} & \Big(\prod_{H\leqslant G} \mathbb{K}\Big)^G \\
\downarrow & & \downarrow & & \downarrow \\
B^A_{\mathbb{K}}(G) & \xrightarrow{m_G} & \Big(\prod_{H\leqslant G} \mathbb{K}H^*\Big)^G & \longrightarrow & \Big(\prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K}\Big)^G
\end{array}$$

of $\mathbb{K}$-algebra homomorphisms, where the left horizontal maps are the mark isomorphisms m_G from (7) given by $(\pi_H \circ \mathrm{res}^G_H)$, the right top horizontal map is the identity, the right bottom horizontal map is the product of the isomorphisms from (4), the middle vertical map is the product of the unique $\mathbb{K}$-algebra homomorphisms $\mathbb{K} \to \mathbb{K}H^*$, and the right vertical map is induced by the G-equivariant map $\mathcal{X}(G) \to \{H \leqslant G\}$, $(H,\Phi) \mapsto H$, between the indexing sets. We denote the primitive idempotents of $B_{\mathbb{K}}(G)$ by e^G_H, for any $H \leqslant G$. Thus, by the above commutative diagram,

$$e^G_G = \sum_{\Phi\in\mathrm{Hom}(G^*,\mathbb{K}^\times)} e^G_{(G,\Phi)}\,. \tag{17}$$

3.4 Lemma *For any $x \in B^A_{\mathbb{K}}(G)$ one has $e^G_G \cdot x = 0$ if and only if $\pi_G(x) = 0$.*

Proof Since $m_G\colon B^A_{\mathbb{K}}(G) \to \prod_{H\leqslant G} \mathbb{K}$ is injective and multiplicative, one has $e^G_G \cdot x = 0$ if and only if $m_G(e^G_G) \cdot m_G(x) = 0$. But $m_G(e^G_G)$ has entry 1 in the G-component and entry 0 everywhere else. Thus, $e^G_G \cdot x = 0$ if and only if the entry of $m(x)$ in the G-component is equal to 0. But this entry equals $\pi_G(x)$. ∎

4 Elementary operations on primitive idempotents

Throughout this section we assume as in Section 3 that G is a finite group such that $S^* = \mathrm{Hom}(S, A)$ is finite for all subquotients S of G, and that $\mathbb{K}$ is a field of characteristic 0 which is a splitting field of S^* for all subquotients S of G.

In this section we will establish formulas for elementary fibered biset operations on the primitive idempotents of $B_{\mathbb{K}}^A(S)$ for subquotients S of G. These formulas will be used in later sections.

4.1 Proposition *Let $H \leqslant G$.*

(a) *For $(L, \Psi) \in \mathcal{X}(H)$ one has $s_{(L,\Psi)}^H \circ \mathrm{res}_H^G = s_{(L,\Psi)}^G : B_{\mathbb{K}}^A(G) \to \mathbb{K}$.*

(b) *For $(K, \Phi) \in \mathcal{X}(G)$ one has*

$$\mathrm{res}_H^G(e_{(K,\Phi)}^G) = \sum_{\substack{(L,\Psi)\in[H\backslash\mathcal{X}(H)] \\ (L,\Psi)=_G(K,\Phi)}} e_{(L,\Psi)}^H \,.$$

(c) *For $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and $H < G$ one has $\mathrm{res}_H^G(e_{(G,\Phi)}^G) = 0$.*

Proof (a) We use the point of view from Remark 2.4. By [BRV19, Equation (13) and Theorem 6.1] the left square in

$$\begin{array}{ccccc}
B_{\mathbb{K}}^A(G) & \xrightarrow{m_G} & \prod_{K\leqslant G} \mathbb{K}K^* & \longrightarrow & \prod_{(K,\Phi)\in\mathcal{X}(G)} \mathbb{K} \\
\Big\downarrow{\scriptstyle \mathrm{res}_H^G} & & \Big\downarrow & & \Big\downarrow \\
B_{\mathbb{K}}^A(H) & \xrightarrow{m_H} & \prod_{L\leqslant H} \mathbb{K}L^* & \longrightarrow & \prod_{(L,\Psi)\in\mathcal{X}(H)} \mathbb{K}
\end{array}$$

is commutative, where the left horizontal maps are the mark homomorphisms from (7), the right horizontal maps are given by the isomorphisms in (4), and the middle and right vertical maps are the canonical projections. Since the right hand square commutes as well, following up with the projection onto the (L, Ψ)-component of $\prod_{(L,\Psi)\in\mathcal{X}(H)} \mathbb{K}$, yields the result.

(b) Since $\mathrm{res}_H^G : B_{\mathbb{K}}^A(G) \to B_{\mathbb{K}}^A(H)$ is a $\mathbb{K}$-algebra homomorphism, $\mathrm{res}_H^G(e_{(K,\Phi)})$ is the sum of certain primitive idempotents $e_{(L,\Psi)}^H$, $(L, \Psi) \in [H\backslash\mathcal{X}(H)]$. Moreover, $e_{(L,\Psi)}^H$ occurs in this sum if and only if $s_{(L,\Psi)}^H(\mathrm{res}_H^G(e_{(K,\Phi)}^G)) \neq 0$. The result follows now immediately from (a).

(c) This follows immediately from Part (b). $\square$

4.2 Proposition *Let $N \trianglelefteq G$.*

(a) *For $(H, \Phi) \in \mathcal{X}(G)$ on has $s_{(H,\Phi)}^G \circ \mathrm{inf}_{G/N}^G = s_{(HN/N,\Phi_N)}^{G/N}$, where $\Phi_N := \Phi \circ \nu^* \circ \alpha^* \in \mathrm{Hom}((HN/N)^*, \mathbb{K}^\times)$ with $\alpha : H/(H \cap N) \xrightarrow{\sim} HN/N$ denoting the canonical isomorphism and $\nu : H \to H/(H \cap N)$ denoting the canonical epimorphism.*

(b) *For $(U/N, \Psi) \in \mathcal{X}(G/N)$ with $N \leqslant U \leqslant G$, one has*

$$\mathrm{inf}_{G/N}^G(e_{(U/N,\Psi)}^{G/N}) = \sum_{\substack{(H,\Phi)\in[G\backslash\mathcal{X}(G)] \\ (HN/N,\Phi_N)=_{G/N}(U/N,\Psi)}} e_{(H,\Phi)}^G \,.$$

Proof (a) We use again the point of view from Remark 2.4. By [BRV19, Equation (12)] applied to $D := \{(g, gN) \mid g \in G\} \leqslant G \times G/N$ and [BRV19, Theorem 6.1] the left square in

$$\begin{array}{ccccc} B_{\mathbb{K}}^A(G/N) & \xrightarrow{m_{G/N}} & \prod\limits_{N \leqslant U \leqslant G} \mathbb{K}(U/N)^* & \longrightarrow & \prod\limits_{(U/N,\Psi) \in \mathcal{X}(G/N)} \mathbb{K} \\ \Big\downarrow \mathrm{inf}_{G/N}^G & & \Big\downarrow & & \Big\downarrow \\ B_{\mathbb{K}}^A(G) & \xrightarrow{m_G} & \prod\limits_{H \leqslant G} \mathbb{K}H^* & \longrightarrow & \prod\limits_{(H,\Phi) \in \mathcal{X}(G)} \mathbb{K} \end{array}$$

is commutative, where the left horizontal maps are the mark homomorphisms from (7), the right horizontal maps are given by the isomorphisms in (4), the middle vertical homomorphism maps the family $(a_{U/N})_{N \leqslant U \leqslant G}$ to $(\mathrm{inf}_{H/(H \cap N)}^H(\alpha^*(a_{HN/N})))_{H \leqslant G}$ with $\alpha \colon H/(H \cap N) \xrightarrow{\sim} HN/N$ denoting the canonical isomorphism, and the right vertical homomorphism maps the family $(a_{(U/N,\Psi)})_{(U/N,\Psi) \in \mathcal{X}(G/N)}$ to the family $(a_{(HN/N,\Phi_N)})_{(H,\Phi) \in \mathcal{X}(G)}$. Since the right hand square commutes as well (note that $\mathrm{inf}_{H/(H \cap N)}^H \colon \mathbb{K}(H/(H \cap N))^* \to \mathbb{K}H^*$ is he $\mathbb{K}$-linear extension of ν^* from (a)), following up with the projection onto the (H, Φ)-component of $\prod_{(H,\Phi) \in \mathcal{X}(G)} \mathbb{K}$, yields the result.

(b) Since $\mathrm{inf}_{G/N}^G \colon B_{\mathbb{K}}^A(G/N) \to B_{\mathbb{K}}^A(G)$ is a $\mathbb{K}$-algebra homomorphism, $\mathrm{inf}_{G/N}^G(e_{(U/N,\Psi)}^{G/N})$ is the sum of certain primitive idempotents $e_{(H,\Phi)}^G$, $(H, \Phi) \in [G \backslash \mathcal{X}(G)]$. Moreover, $e_{(H,\Phi)}^G$ occurs in this sum if and only if $s_{(H,\Phi)}^G(\mathrm{inf}_{G/N}^G(e_{(U/N,\Psi)}^{G/N})) \neq 0$. Part (a) now implies the result. □

4.3 Proposition *Let $N \trianglelefteq G$.*

(a) *For all $(H, \Phi) \in \mathcal{X}(G)$, there exists $m_{(H,\Phi)}^{G,N} \in \mathbb{K}$ such that*

$$\mathrm{def}_{G/N}^G(e_{(H,\Phi)}^G) = m_{(H,\Phi)}^{G,N} \cdot e_{(HN/N,\Phi_N)}^G \tag{18}$$

with Φ_N defined as in Proposition 4.2(a).

(b) *For all $\Phi \in G^*$ one has*

$$m_{(G,\Phi)}^N := m_{(G,\Phi)}^{G,N} = \frac{|(G/N)^*|}{|G| \cdot |G^*|} \sum_{\substack{K \leqslant G \\ KN = G \\ \Phi|_{K^\perp} = 1}} |K| \cdot |K^\perp| \cdot \mu(K, G) \in \mathbb{Q}\,. \tag{19}$$

Proof (a) For any $x \in B_{\mathbb{K}}^A(G/N)$ we have

$$\begin{aligned} x \cdot \mathrm{def}_{G/N}^G(e_{(H,\Phi)}^G) &= \mathrm{def}_{G/N}^G(\mathrm{inf}_{G/N}^G(x) \cdot e_{(H,\Phi)}^G) = \mathrm{def}_{G/N}^G(s_{(H,\Phi)}^G(\mathrm{inf}_{G/N}^G(x)) \cdot e_{(H,\Phi)}^G) \\ &= \mathrm{def}_{G/N}^G(s_{(HN/N,\Phi_N)}^{G/N}(x) \cdot e_{(H,\Phi)}^G) = s_{(HN/N,\Phi_N)}^{G/N}(x) \cdot \mathrm{def}_{G/N}^G(e_{(H,\Phi)}^G)\,. \end{aligned}$$

In fact, the first equation follows from the Green biset functor axioms (see [BRV19, Definition 7.2(a)] and [R11, Definición 3.2.7, Lema 4.2.3]), the second from (8), and the third from Proposition 4.2(a). Choosing $x = e_{(HN/N,\Phi_N)}^{G/N}$, and reading the above equations backward, we obtain

$$\mathrm{def}_{G/N}^G(e_{(H,\Phi)}^G) = \mathrm{def}_{G/N}^G(e_{(H,\Phi)}^G) \cdot e_{(HN/N,\Phi_N)}^G\,.$$

Now, (8) implies the result with $m^G_{(H,\Phi)} = s^{G/N}_{(HN/N,\Phi_N)}(\mathrm{def}^G_{G/N}(e^G_{(H,\Phi)}))$.

(b) Substituting the explicit idempotent formula (11) for $e^G_{(G,\Phi)}$ and using the explicit formula for $\mathrm{def}^G_{G/N}\colon B^G_{\mathbb{K}}(G) \to B^A_{\mathbb{K}}(G/N)$ from 2.2(c), the left hand side of (18) is equal to

$$\frac{1}{|G|\cdot|G^*|} \sum_{\substack{K\leqslant G\\ \Phi|_{K^\perp}=1}} \sum_{\substack{\phi\in G^*\\ \phi|_{K\cap N}=1}} |K|\,\mu(K,G)\,\Phi(\phi^{-1})\,[KN/K,\widetilde{\phi|_K}]_{G/N}\,,$$

where $\widetilde{\phi|_K}(kN) := \phi(k)$ for $k\in K$. Moreover, using the explicit formula (11) for $e^{G/N}_{(G/N,\Phi_N)}$, the right hand side of (18) is equal to

$$\frac{m^N_{(G,\Phi)}}{|G/N|\cdot|(G/N)^*|} \sum_{\substack{U/N\leqslant G/N\\ \Phi_N|_{(U/N)^\perp}=1}} \sum_{\psi\in(G/N)^*} |U/N|\,\mu(U/N,G/N)\,\Phi_N(\psi^{-1})\,[U/N,\psi|_{U/N}]_{G/N}\,.$$

Next we compare the coefficients at the standard basis element $[G/N,1]_{G/N}$ of $B^A_{\mathbb{K}}(G/N)$ on both sides. On the left hand side, we only have to sum over those $K\leqslant G$ with $KN=G$ and those $\phi\in G^*$ with $\widetilde{\phi|_K}=1$. By the definition of $\widetilde{\phi|_K}$, this is equivalent to $\phi\in K^\perp$. But then $\Phi|_{K^\perp}=1$ implies that $\Phi(\phi^{-1})=1$ for all such ϕ. Thus, the coefficient of $[G/N,1]_{G/N}$ on the left hand side of (18) is equal to

$$\frac{1}{|G|\cdot|G^*|} \sum_{\substack{K\leqslant G\\ \Phi|_{K^\perp}=1\\ KN=G}} |K|\,|K^\perp|\,\mu(K,G)\,.$$

On the right hand side of (18) only the summands with $U=G$ and $\psi=1$ contribute to the coefficient of $[G/N,1]_{G/N}$ and this coefficient evaluates to $m^N_{(G,\Phi)}/|(G/N)^*|$. The result follows. □

4.4 Proposition *Let H be a subgroup of G and let $(K,\Psi)\in\mathcal{X}(H)$. Then*

$$\mathrm{ind}^G_H(e^H_{(K,\Psi)}) = \frac{|N_G(K,\Psi)|}{|N_H(K,\Psi)|}\cdot e^G_{(K,\Psi)}\,.$$

Proof This is an immediate consequence of the explicit formula in (11), since $\mathrm{ind}^G_H([L,\phi]_H) = [L,\phi]_G$ for any $(L,\phi)\in\mathcal{M}(H)$. □

4.5 Proposition (a) *For every isomorphism $f\colon G \xrightarrow{\sim} G'$ and $(H,\Phi)\in\mathcal{X}(G)$ one has $\mathrm{iso}_f(e^G_{(H,\Phi)}) = e^{G'}_{(f(H),\Phi\circ(f|_H)^*)}$.*

(b) *For every $g\in G$, $H\leqslant G$, and $(K,\Psi)\in\mathcal{X}(H)$ one has ${}^g e^H_{(K,\Psi)} = e^{{}^gH}_{({}^gK,{}^g\Psi)}$.*

(c) *For every $(H,\Phi)\in\mathcal{X}(G)$ and $\alpha\in G^*$ one has $\mathrm{tw}_\alpha(e^G_{(H,\Phi)}) = \Phi(\alpha|_H)\,e^G_{(H,\Phi)}$ and $\Delta(e^G_{(H,\Phi)})\cdot_G \mathrm{tw}_\alpha = \Phi(\alpha|_H)\,\Delta(e^G_{(H,\Phi)})$, with Δ as in 2.1(c).*

Proof All three parts follow immediately from the explicit formulas for the three operations in 2.2(c), the explicit idempotent formula (11), and the formula in (3). □

5 Three lemmata

Throughout this section, G and H denote finite groups and $\mathbb{K}$ a field of characteristic 0 such that S^* is finite and $\mathbb{K}$ is a splitting field of S^* for all subquotients S of G and H.

For $U \leqslant G \times H$ we set $q(U) := U/(k_1(U) \times k_2(U)$. Thus, $q(U) \cong p_i(U)/k_i(U)$ for $i = 1, 2$.

5.1 Lemma *Let G and H be finite groups and let k be a commutative ring. For $(U, \phi) \in \mathcal{M}(G\times H)$ with $p_1(U) = G$ and $p_2(U) = H$ the following are equivalent:*

(i) *There exists $\alpha \in G^*$ such that $\alpha|_{k_1(U)} = \phi_1$.*

(ii) *There exists $\beta \in H^*$ such that $\beta|_{k_2(U)} = \phi_2$.*

(iii) *There exists $\psi \in (G \times H)^*$ such that $\psi|_U = \phi$.*

(iv) *In the category $\mathcal{C}^A$, the morphism $\left[\frac{G\times H}{U,\phi}\right]$ factors through $q(U)$.*

(v) *In the category $\mathcal{C}^A_k$, the morphism $\left[\frac{G\times H}{U,\phi}\right]$ factors through $q(U)$.*

Proof Clearly, (iii) implies (i) and (ii).

We next show that (i) implies (iii). Let $\alpha \in G^*$ be an extension of ϕ_1. Since $\phi \in U^*$ and $\alpha \times 1 \in (G \times \{1\})^*$ coincide on $U \cap (G \times \{1\}) = k_1(U) \times \{1\}$ and since $G \times H = (G \times \{1\})U$ with $G \times \{1\}$ normal in $G \times H$, the function $\psi\colon G \times H \to A$, $(g, 1)u \mapsto \alpha(g)\phi(u)$ is well-defined and extends ϕ. It is also a homomorphism, since $G \times \{1\}$ is normal in $G \times H$ and $\alpha \times 1$ is U-stable.

Similarly one proves that (ii) implies (iii).

Next we show that (iii) and implies (iv). Let $\psi = \alpha \times \beta \in (G \times H)^*$ extend $\phi \in U^*$. By (3), we have

$$\left[\frac{G \times H}{U, \phi}\right] = \mathrm{tw}_\alpha \underset{G}{\cdot} \left[\frac{G \times H}{U, 1}\right] \underset{H}{\cdot} \mathrm{tw}_\beta$$

and the morphism $\left[\frac{G\times H}{U,1}\right]$ factors through $G/k_1(U)$ by Theorem 2.3.

Clearly, (iv) implies (v).

Finally, we show that (v) implies (iii). Assume that $\left[\frac{G\times H}{U,\phi}\right]$ factors in $\mathcal{C}^A_k$ through $K := q(U)$ with $K \cong G/k_1(U)$. Then there exist $(V, \Psi) \in \mathcal{M}(G \times K)$ and $(W, \rho) \in \mathcal{M}(K \times H)$ such that $\left[\frac{G\times H}{U,\phi}\right]$ occurs with nonzero coefficient in $\left[\frac{G\times K}{V,\psi}\right] \cdot_K \left[\frac{K\times H}{W,\rho}\right]$. By the formula in (3), this implies that there exists $t \in K$ such that $\psi_2|_{K_t} = \rho_1|_{K_t}$ with $K_t := k_2(V) \cap {}^t k_1(W)$ and $(U, \phi) = {}^{(g,h)}(V * {}^{(t,1)}W, \psi * {}^{(t,1)}\rho)$. Replacing (V, ψ) with ${}^{(g,1)}(V, \psi)$ and (W, ρ) with ${}^{(t,h)}(W, \rho)$, we may assume that there exist $(V, \psi) \in \mathcal{M}(G \times K)$ and $(W, \rho) \in \mathcal{M}(K \times H)$ with $\psi_2|_{k_2(V)\cap k_1(W)} = \rho_1|_{k_2(V)\cap k_1(W)}$ and $(V * W, \psi * \rho) = (U, \phi)$. By [Bc10, 2.3.22.2] one has

$$k_1(V) \leqslant k_1(V * W) = k_1(U) \leqslant p_1(U) \leqslant p_1(V)\,.$$

Since $p_1(U) = G$, this implies that $p_1(V) = G$. Moreover, since $p_1(U)/k_1(U)$ is isomorphic to a subquotient of $p_1(V)/k_1(V) \cong p_2(V)/k_2(V)$, which in turn is a subquotient of $K \cong p_1(U)/k_1(U)$, we otain that $k_1(V) = k_1(U)$, $p_2(V) = K$ and $k_2(V) = \{1\}$. Since $\psi_2 = 1$, it extends trivially to $K = p_2(V)$. By the first part of the proof (note that $p_1(V) = G$ and $p_2(V) = K$) we also obtain that ψ extends to $\alpha \times 1 \in (G \times K)^*$ for some $\alpha \in G^*$. Similarly one shows that ρ extends to

$1 \times \beta \in (K \times H)^*$ for some $\beta \in H^*$. But then $\phi = \psi * \rho$ is the restriction of $(\alpha \times 1) * (1 \times \beta) = \alpha \times \beta$ and (iii) holds. □

5.2 Lemma *Let $(U, \phi) \in \mathcal{M}(G \times H)$.*

(a) *If $\Phi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$ satisfies $\left[\frac{G\times H}{U,\phi}\right] \cdot_H e^H_{(H,\Phi)} \neq 0$ then $p_2(U) = H$ and ϕ_2 extends to H^*.*

(b) *If $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ satisfies $e^G_{(G,\Phi)} \cdot_G \left[\frac{G\times H}{U,\phi}\right] \neq 0$ then $p_1(U) = G$ and ϕ_1 extends to G^*.*

Proof We only prove Part (a). Part (b) is proved similarly.

Since $\left[\frac{G\times H}{U,\phi}\right] = \left[\frac{G\times p_2(U)}{U,\phi}\right] \cdot_{p_2(U)} \mathrm{res}^H_{p_2(U)}$, Proposition 4.1(c) implies that $p_2(U) = H$.

We will show that ϕ_2 extends to H by induction on $|G|$. If $|G| = 1$ then ϕ_1 is trivial, thus extends to G, and Lemma 5.1 implies that ϕ_2 extends to H. From now on assume that $|G| > 1$. We distinguish two cases.

Case 1: $\pi_G\left(\left[\frac{G\times H}{U,\phi}\right] \cdot_H e^H_{(H,\Phi)}\right) = 0$. By Lemma 3.4, this implies that $e^G_G \cdot \left(\left[\frac{G\times H}{U,\phi}\right] \cdot_H e^H_{(H,\Phi)}\right) = 0$ and therefore, using the primitive idempotents e^G_K of the Burnside ring (see Remark 3.3) and [CY19, Proposition 2], we have

$$
\begin{aligned}
0 \neq \left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_H e^H_{(H,\Phi)} &= (1 - e^G_G) \cdot \left(\left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_H e^H_{(H,\Phi)}\right) \\
&= \sum_{\substack{K\in[G\backslash\mathcal{S}(G)] \\ K<G}} e^G_K \cdot \left(\left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_H e^H_{(H,\Phi)}\right) = \sum_{\substack{K\in[G\backslash\mathcal{S}(G)] \\ K<G}} \Delta(e^G_K) \mathop{\cdot}_G \left(\left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_H e^H_{(H,\Phi)}\right) .
\end{aligned}
$$

Moreover, using the explicit formula for e^G_K (in the special case that A is trivial), we have

$$
\begin{aligned}
0 \neq \left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_H e^H_{(H,\Phi)} &\in \sum_{K<G} \mathbb{K} \cdot \left[\frac{G\times G}{\Delta(K),1}\right] \mathop{\cdot}_G \left(\left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_H e^H_{(H,\Phi)}\right) \\
&= \sum_{K<G} \mathbb{K} \cdot \left(\left[\frac{G\times G}{\Delta(K),1}\right] \mathop{\cdot}_G \left[\frac{G\times H}{U,\phi}\right]\right) \mathop{\cdot}_H e^H_{(H,\Phi)} = \sum_{K<G} \mathbb{K} \cdot \left[\frac{G\times H}{\Delta(K)*U, 1*\phi}\right] \mathop{\cdot}_H e^H_{(H,\Phi)} .
\end{aligned}
$$

Therefore, there exists $K < G$ such that $\left[\frac{G\times H}{\Delta(K)*U,1*\phi}\right] \cdot_H e^H_{(H,\Phi)} \neq 0$. Note that $p_1(\Delta(K)*U) \leqslant K$, so that we can decompose

$$
\left[\frac{G\times H}{\Delta(K)*U, 1*\phi}\right] = \mathrm{ind}^G_K \mathop{\cdot}_K \left[\frac{K\times H}{\Delta(K)*U, 1*\phi}\right] .
$$

Thus,

$$
\left[\frac{K\times H}{\Delta(K)*U, 1*\phi}\right] \mathop{\cdot}_H e^H_{(H,\Phi)} \neq 0 . \tag{20}
$$

By the first part of the proof this implies that $p_2(\Delta(K)*U) = H$. Moreover, it is straightforward to verify that $k_2(\Delta(K)*U) = k_2(U)$ and that $(1*\phi)_2 = \phi_2 \in k_2(U)^*$. Since $K < G$, the inductive hypothesis applied to (20) yields that ϕ_2 extends to H.

Case 2: $\pi_G\Big(\Big[\frac{G\times H}{U,\phi}\Big]\cdot_H e^H_{(H,\Phi)}\Big)\neq 0$. The explicit formula for $e^H_{(H,\Phi)}$ in (11) implies that

$$\sum_{\substack{K\leqslant H\\ \Phi|_{K^\perp}=1}}\ \sum_{\psi\in H^*}|K|\,\mu(K,H)\,\Phi(\psi^{-1})\,\pi_G\bigg(\bigg[\frac{G\times H}{U,\phi}\bigg]\underset{H}{\cdot}[K,\psi|_K]_H\bigg)\neq 0\,.$$

Thus, there exists $K\leqslant H$ and $\psi\in H^*$ such that $\pi_G\left(\Big[\frac{G\times H}{U,\phi}\Big]\cdot_H [K,\psi|_K]_H\right)\neq 0$. This implies that $U*K=G$ and $\phi_2|_{k_2(U)\cap K}=\psi|_{k_2(U)\cap K}$. Since ϕ_2 is stable under $p_2(U)=H$ and ϕ_2 and ψ coincide on $k_2(U)\cap K$, there exists an extension $\beta\in(k_2(U)K)^*$ of ϕ_2 and ψ. Moreover, $U*K=G$ implies $k_2(U)K=H$. In fact, if $h\in H=p_2(U)$ then there exists $g\in G$ with $(g,h)\in U$. Since $g\in G=U*K$, there exists $k\in K$ such that $(g,k)\in U$. But then $hk^{-1}\in k_2(U)$ and $h\in k_2(U)K$. This completes the lemma. □

5.3 Lemma *Let $(U,\phi)\in\mathcal{M}(G\times H)$, $\Phi\in\mathrm{Hom}(G^*,\mathbb{K}^\times)$ and $\Psi\in\mathrm{Hom}(H^*,\mathbb{K}^\times)$. Then*

$$e^G_{(G,\Phi)}\cdot\bigg(\bigg[\frac{G\times H}{U,\phi}\bigg]\underset{H}{\cdot}e^H_{(H,\Psi)}\bigg)=0$$

unless $p_1(U)=G$, $p_2(U)=H$, ϕ has an extension $\alpha\times\beta\in(G\times H)^*$, $\Phi_{k_1(U)}=\Psi_{k_2(U)}\circ\eta_U^*$, and $m^{k_2(U)}_{(H,\Phi)}\neq 0$, in which case one has

$$e^G_{(G,\Phi)}\cdot\bigg(\bigg[\frac{G\times H}{U,\phi}\bigg]\underset{H}{\cdot}e^H_{(H,\Psi)}\bigg)=\Phi(\alpha)\,\Psi(\beta)\,m^{k_2(U)}_{(H,\Psi)}\cdot e^G_{(G,\Phi)}\,.$$

Proof Assume that

$$e^G_{(G,\Phi)}\cdot\bigg(\bigg[\frac{G\times H}{U,\phi}\bigg]\underset{H}{\cdot}e^H_{(H,\Psi)}\bigg)\neq 0 \tag{21}$$

By Lemma 5.2, $p_2(U)=H$ and ϕ_2 extends to H. Assume $p_1(U)<G$. Then $\Big[\frac{G\times H}{U,\phi}\Big]\cdot e^H_{(H,\Psi)}$ is in the $\mathbb{K}$-span of elements of the form $[U*L,\psi]_G$, with $L\leqslant H$ and $\psi\in(U*L)^*$. Since $U*L\leqslant p_1(U)<G$ we obtain $m_G(e_{(G,\Phi)}\cdot[U*L,\psi]_G)=m_G(e_{(G,\Phi)})\cdot m_G([U*L,\psi]_G)=0$, because the first factor vanishes in the components indexed by proper subgroups of G and the second factor vanishes in the G-component. By the injectivity of m_G this implies that the element in (21) vanishes, a contradiction. Thus, $p_1(U)=G$. By Lemma 5.1, ϕ has an extension $\alpha\times\beta\in(G\times H)^*$. Further,

$$\begin{aligned}e^G_{(G,\Phi)}\cdot\bigg(\bigg[\frac{G\times H}{U,\phi}\bigg]\underset{H}{\cdot}e^H_{(H,\Psi)}\bigg)&=e^G_{(G,\Phi)}\cdot\bigg(\mathrm{tw}_\alpha\underset{G}{\cdot}\bigg[\frac{G\times H}{U,1}\bigg]\underset{H}{\cdot}\mathrm{tw}_\beta\underset{H}{\cdot}e^H_{(H,\Psi)}\bigg)\\&=\Delta(e^G_{(G,\Phi)})\underset{G}{\cdot}\mathrm{tw}_\alpha\underset{G}{\cdot}\bigg[\frac{G\times H}{U,1}\bigg]\underset{H}{\cdot}\mathrm{tw}_\beta\underset{H}{\cdot}e^H_{(H,\Psi)}\,,\end{aligned}$$

where the last equation follows from [CY19, Proposition 2]. By Proposition 4.5(c) we have $\mathrm{tw}_\beta\cdot_H e^H_{(H,\Psi)}=\mathrm{tw}_\beta(e^H_{(H,\Psi)})=\Psi(\beta)\cdot e^H_{(H,\Psi)}$ and $\Delta(e^G_{(G,\Phi)})\cdot_G\mathrm{tw}_\alpha=\Phi(\alpha)\cdot\Delta(e^G_{(G,\Phi)})$. Thus, using again

[CY19, Proposition 2] and Theorem 2.3,

$$
\begin{aligned}
e^G_{(G,\Phi)} \cdot \left(\left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_{H} e^H_{(H,\Psi)} \right) &= \Phi(\alpha)\Psi(\beta)\cdot \Delta(e^G_{(G,\Phi)}) \mathop{\cdot}_{G} \left[\frac{G\times H}{U,1}\right] \mathop{\cdot}_{H} e^H_{(H,\Psi)} \\
&= \Phi(\alpha)\,\Psi(\beta)\, e_{(G,\Phi)} \cdot \left(\mathrm{inf}^G_{G/k_1(U)} \mathop{\cdot}_{G/k_1(U)} \left[\frac{G\times H}{\bar{U},1}\right]_{H/k_2(U)} \mathop{\cdot}\, \mathrm{def}^H_{H/k_2(U)} \mathop{\cdot}_{H} e^H_{(H,\Psi)} \right) \\
&= \Phi(\alpha)\,\Psi(\beta)\, e_{(G,\Phi)} \cdot \left(\mathrm{inf}^G_{G/k_1(U)} \mathop{\cdot}_{G/k_1(U)} \mathrm{iso}_{\eta_U} \mathop{\cdot}_{H/k_2(U)} \mathrm{def}^H_{H/k_2(U)} \mathop{\cdot}_{H} e^H_{(H,\Psi)} \right)
\end{aligned}
$$

with $\eta_U\colon H/k_2(U) \xrightarrow{\sim} G/k_1(U)$ the isomorphism induced by U (see [Bc10, Lemma 2.3.25]). Using Propositions 4.3(a), 4.5(a) and 4.2(b), we obtain

$$
e^G_{(G,\Phi)} \cdot \left(\left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_{H} e^H_{(H,\Psi)} \right) = \Phi(\alpha)\,\Psi(\beta)\, m^{k_2(U)}_{(H,\Psi)} \sum_{(K,\Theta)} e^G_{(G,\Phi)} \cdot e^G_{(K,\Theta)}\,, \tag{22}
$$

where the sum runs over those $(K,\Theta) \in [G\backslash\mathcal{X}(G)]$ satisfying $(Kk_1(U)/k_1(U), \Theta_{k_1(U)}) =_{G/k_1(U)} (G/k_1(U), \Psi_{k_2(U)} \circ \eta_U^*)$. Since the term in (22) is nonzero, one of these pairs (K,Θ) must be G-conjugate, and then also equal, to (G,Φ). This implies the result. □

6 The constant $m^N_{(G,\Phi)}$

Throughout this section, G and H denote finite groups and $\mathbb{K}$ denotes a field of characteristic 0 such that for any subquotients S of G and T of H the groups S^* and T^* are finite and $\mathbb{K}$ is a splitting field for S^* and T^*.

In this section we prove the crucial Proposition 6.4 stating that $m^M_{(G,\Phi)} = m^N_{(G,\Phi)}$ if $(G/M, \Phi_M) \cong (G/N, \Phi_N)$ (see Definition 7.3(a)).

6.1 Proposition *Let N and M be normal subgroups of G with $N \leqslant M$ and let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$. Then*

$$
m^M_{(G,\Phi)} = m^N_{(G,\Phi)} \cdot m^{M/N}_{(G/N,\Phi_N)}\,.
$$

Proof This follows immediately from Proposition 4.3(a) and applying $\mathrm{iso}_f \circ \mathrm{def}^{G/N}_{(G/N)/(M/N)} \circ \mathrm{def}^G_{G/N} = \mathrm{def}^G_{G/M}$ to $e^G_{(G,\Phi)}$, where $f\colon (G/N)/(M/N) \to G/M$ is the canonical isomorphism. □

6.2 Lemma *Let $f_1, f_2\colon G \to H$ be group homomorphisms, let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$, and let $K \leqslant G$ be such that $\Phi|_{K^\perp} = 1$ and $f_1|_K = f_2|_K$. Then $\Phi \circ f_1^* = \Phi \circ f_2^* \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$.*

Proof Let $\lambda \in H^*$. Then $(\Phi\circ f_1^*)(\lambda) = (\Phi\circ f_2^*)(\lambda)$ if and only if $\Phi(\lambda\circ f_1) = \Phi(\lambda\circ f_2)$ which in turn is equivalent to $\Phi((\lambda\circ f_1)\cdot(\lambda\circ f_2)^{-1}) = 1$. But, since $f_1|_K = f_2|_K$, we have $(\lambda\circ f_1)\cdot(\lambda\circ f_2)^{-1} \in K^\perp$ and since $\Phi|_{K^\perp} = 1$ the result follows. □

6.3 Proposition *For $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and normal subgroups M and N of G one has*

$$m^M_{(G,\Phi)} = \frac{1}{|G|\cdot|G^*|} \sum_{\substack{K\leqslant G\\ KN=KM=G\\ \Phi|_{K^\perp}=1}} |K|\,\mu(K,G)\,|\Sigma^K_{M,N}|\, m^{(K\cap M)N/N}_{(G/N,\Phi_N)}\,,$$

where $\Sigma^K_{M,N}$ is the set of elements $\phi \in G^*$ such that $\phi|_{K\cap M\cap N} = 1$ and such that there exists $\psi \in ((G/M)\times(G/N))^*$ with $\psi(kM, kN) = \phi(k)$ for all $k \in K$.

Proof Consider the element

$$v := e^{G/M}_{(G/M,\Phi_M)} \cdot \left(\mathrm{def}^G_{G/M} \underset{G}{\cdot} \Delta(e^G_{(G,\Phi)}) \underset{G}{\cdot} \mathrm{inf}^G_{G/N} \underset{G/N}{\cdot} e^{G/N}_{(G/N,\Phi_N)}\right) \in B^A_{\mathbb{K}}(G/M)\,.$$

Then, on the one hand,

$$v = m^M_{(G,\Phi)} e^{G/M}_{(G/M,\Phi_M)} \in B^A_{\mathbb{K}}(G/M)\,. \tag{23}$$

In fact, by [CY19, Proposition 2] and Proposition 4.2(b),

$$\Delta(e^G_{(G,\Phi)}) \underset{G}{\cdot} \mathrm{inf}^G_{G/N} \underset{G/N}{\cdot} e^{G/N}_{(G/N,\Phi_N)} = e^G_{(G,\Phi)} \cdot (\mathrm{inf}^G_{G/N} \underset{G/N}{\cdot} e^{G/N}_{(G/N,\Phi_N)}) = e^G_{(G,\Phi)}\,,$$

and then Proposition 4.3(a) implies (23). On the other hand, using the formula in (11) for $e^G_{(G,\Phi)}$ we obtain

$$v = \frac{1}{|G|\cdot|G^*|} \sum_{\substack{K\leqslant G\\ \Phi|_{K^\perp}=1}} \sum_{\phi\in G^*} |K|\,\mu(K,G)\,\Phi(\phi^{-1})\, e^{G/M}_{(G/M,\Phi_M)} \cdot \left(x_{K,\phi} \underset{G/N}{\cdot} e^{G/N}_{(G/N,\Phi_N)}\right) \tag{24}$$

with

$$x_{K,\phi} := \mathrm{def}^G_{G/M} \underset{G}{\cdot} \Delta([K,\phi|_K]_G) \underset{G}{\cdot} \mathrm{inf}^G_{G/N} = \begin{cases} \left[\frac{G/M\times G/N}{\Delta^K_{M,N},\bar\phi}\right], & \text{if } \phi|_{K\cap M\cap N} = 1,\\ 0, & \text{otherwise,}\end{cases}$$

for $K \leqslant G$ with $\Phi|_{K^\perp} = 1$, by the formula in (3), where, in the first case, $\Delta^K_{M,N} := \{(kM, kN) \mid k \in K\}$ and $\bar\phi((kM, kN) := \phi(k)$ for $k \in K$. Note that $A_K := k_1(\Delta^K_{M,N}) = (K\cap N)M/M$, $B_K := k_2(\Delta^K_{M,N}) = (K\cap M)N/N$, $p_1(\Delta^K_{M,N}) = KM/M$, and $p_2(\Delta^K_{M,N}) = KN/N$. Lemma 5.3 implies that, if the term $e^{G/M}_{(G/M,\Phi_M)} \cdot (x_{K,\phi} \underset{G/N}{\cdot} e^{G/N}_{(G/N,\Phi_N)})$ in (24) is nonzero then $KM = G = KN$, $\phi|_{K\cap M\cap N} = 1$, and $\bar\phi$ extends to some $\psi = \alpha\times\beta \in ((G/M)\times(G/N))^*$. Then the formula in Lemma 5.3 yields

$$v = \frac{1}{|G|\cdot|G^*|} \sum_{\substack{K\leqslant G\\ KM=G=KN\\ \Phi|_{K^\perp}=1}} \sum_{\phi\in\Sigma^K_{M,N}} |K|\,\mu(K,G)\,\Phi(\phi^{-1})\,\Phi_M(\alpha_{K,\phi})\,\Phi_N(\beta_{K,\phi})\, m^{B_K}_{(G/N,\Phi_N)}\cdot$$
$$\cdot\, e^{G/M}_{(G/M,\Phi_M)}\,, \tag{25}$$

where, for K and ϕ as above, $\alpha_{K,\phi} \in (G/M)^*$ and $\beta_{K,\psi} \in (G/N)^*$ are chosen such that $\alpha_{K,\phi}\times\beta_{K,\phi}$ is an extension of $\bar\phi$. Denoting the inflations of $\alpha_{K,\psi}$ and $\beta_{K,\psi}$ to G by $\tilde\alpha_{K,\psi} \in$ and $\tilde\beta_{K,\psi}$, we have

$$\Phi(\phi^{-1})\Phi_M(\alpha_{K,\phi})\Phi_N(\beta_{K,\phi}) = \Phi(\phi^{-1}\tilde\alpha_{K,\phi}\tilde\beta_{K,\phi}) = 1$$

since $\tilde\alpha_{K,\phi}(k)\tilde\beta_{K,\phi}(k) = \alpha_{K,\phi}(kM)\beta_{K,\phi}(kN) = \bar\phi(kM,kN) = \phi(k)$ for all $k \in K$ and $\Phi|_{K^\perp} = 1$. Thus,

$$v = \frac{1}{|G|\cdot|G^*|}\sum_{\substack{K\leqslant G\\ KM=G=KN\\ \Phi|_{K^\perp}=1}} |\Sigma_{M,N}^K|\,|K|\,\mu(K,G)\,m_{(G/N,\Phi_N)}^{B_K}\,e_{(G/M,\Phi_M)}^{G/M}\,. \tag{26}$$

Comparing Equations (23) and (26), the formula for $m_{(G,\Phi)}^M$ follows. ∎

6.4 Proposition *Let M and N be normal subgroups of G such that there exists an isomorphism $f\colon G/N \xrightarrow{\sim} G/M$ and let $\Phi \in \mathrm{Hom}(G^*,\mathbb{K}^\times)$ be such that $\Phi_N \circ f^* = \Phi_M \in \mathrm{Hom}((G/M)^*,\mathbb{K}^\times)$. Then one has $m_{(G,\Phi)}^M = m_{(G,\Phi)}^N$.*

Proof We proceed by induction on $|G|$. If $|G| = 1$ the result is clearly true. So assume that $|G| > 1$. If $M = N$ is the trivial subgroup of G then the result holds for trivial reasons. So assume that M and N are not trivial. By Proposition 6.3 one has

$$m_{(G,\Phi)}^M = \frac{1}{|G|\cdot|G^*|}\sum_{\substack{K\leqslant G\\ KN=KM=G\\ \Phi|_{K^\perp}=1}} |K|\,\mu(K,G)\,|\Sigma_{M,N}^K|\,m_{(G/N,\Phi_N)}^{(K\cap M)N/N}$$

and

$$m_{(G,\Phi)}^N = \frac{1}{|G|\cdot|G^*|}\sum_{\substack{K\leqslant G\\ KN=KM=G\\ \Phi|_{K^\perp}=1}} |K|\,\mu(K,G)\,|\Sigma_{N,M}^K|\,m_{(G/M,\Phi_M)}^{(K\cap N)M/M}\,.$$

We will show that these sums coincide by comparing them summand by summand. Since $\Sigma_{M,N}^K = \Sigma_{N,M}^K$, it suffices to show that $m_{(G/N,\Phi_N)}^{(K\cap M)N/N} = m_{(G/M,\Phi_M)}^{(K\cap N)M/M}$. By Proposition 4.3(b), for any $X \trianglelefteq G/N$ we have

$$m_{(G/N,\Phi_N)}^X = \frac{|((G/N)/X)^*|}{|G/N|\cdot|(G/N)^*|}\sum_{\substack{L\leqslant G/N\\ LX=G/N\\ (\Phi_N)|_{L^\perp}=1}} |L|\,|L^\perp||\,\mu(L,G/N)$$

and

$$m_{(G/M,\Phi_M)}^{f(X)} = \frac{|((G/M)/\alpha(X))^*|}{|G/M|\cdot|(G/M)^*|}\sum_{\substack{K\leqslant G/N\\ Kf(X)=G/N\\ \Phi_M|_{K^\perp}=1}} |K|\,|K^\perp|\,\mu(K,G/M)\,.$$

Note that the summand for L in the first sum is equal to the summand for $K = f(L)$ in the second sum. Thus, with $X = (K\cap M)N/N$, we obtain

$$m_{(G/N,\Phi_N)}^{(K\cap M)N/N} = m_{(G/M,\Phi_M)}^{f((K\cap M)N/N)} \tag{27}$$

for any $K \leqslant G$ with $KM = G = KN$ and $\Phi|_{K^\perp} = 1$. It now suffices to find an isomorphism $f_1\colon (G/M)/f((K\cap M)N/N) \xrightarrow{\sim} (G/M)/((K\cap N)M/M)$ such that $(\Phi_M)_{f((K\cap M)N/N)} \circ f_1^* = (\Phi_M)_{(K\cap N)M/M}$, since then, by induction, we have $m_{(G/M,\Phi_M)}^{f((K\cap M)N/N)} = m_{(G/M,\Phi_M)}^{(K\cap N)M/M}$ and together with Equation (27) this implies that desired equation. We define $f_1 := \eta \circ \bar{f}^{-1}$, where $\bar{f}\colon (G/N)/((K\cap M)N/N) \xrightarrow{\sim} (G/M)/f((K\cap M)N/N)$ is induced by f and $\eta := \eta_{\Delta^K_{M,N}}\colon (G/N)/((K\cap M)N/N) \xrightarrow{\sim} (G/M)/((K\cap N)M/M)$ is induced by $\Delta^K_{M,N} := \{(kM, kN) \mid k \in K\}$, see [Bc10, Lemma 2.3.25]. It remains to prove that

$$\Phi \circ \nu_M^* \circ \nu_{f((K\cap M)N/N)}^* \circ f_1^* = \Phi \circ \nu_M^* \circ \nu_{((K\cap N)M/M)}^* \,, \tag{28}$$

where $\nu_M\colon G \to G/M$, $\nu_{f(K\cap M)N/N}\colon G/M \to (G/M)/f((K\cap M)N/N)$, and $\nu_{(K\cap N)M/M}\colon G/M \to (G/M)/((K\cap N)M/M)$ denote the natural epimorphisms. But $f_1^* = (\bar{f}^{-1})^* \circ \eta^*$, $\nu^*_{f((K\cap M)N/N)} \circ (\bar{f}^{-1})^* = (f^{-1})^* \circ \nu^*_{(K\cap N)M/M}$, and $\Phi \circ \nu_M^* \circ (f^{-1})^* = \Phi_M \circ (f^{-1})^* = \Phi_N = \Phi \circ \nu_N^*$. Thus, the left hand side of Equation (28) is equal to $\Phi \circ \nu_N^* \circ \nu^*_{(K\cap M)N/N} \circ \eta^*$. By Lemma 6.2 and since $\Phi|_{K^\perp} = 1$, it now suffices to show that

$$(\eta \circ \nu_{(K\cap M)N/N} \circ \nu_N)|_K = (\nu_{(K\cap M)N/N} \circ \nu_N)|_K \,.$$

But this follows from $(kM, kN) \in \Delta^K_{M,N}$ for $k \in K$, and the proof is complete. ∎

7 B^A-pairs and the subfunctors $E^N_{(G,\Phi)}$ of $B^A_{\mathbb{K}}$

Throughout this section G denotes a finite group and we assume that $\mathbb{K}$ is a field of characteristic 0 which is a splitting field for $\mathbb{K}G^*$ for all finite groups G. This is equivalent to requiring that, for any torsion element a of A, the field $\mathbb{K}$ has a root of unity whose order is the order of a.

In this section we introduce the important subfunctors $E_{(G,\Phi)}$ of $B^A_{\mathbb{K}}$ and study their properties.

7.1 Definition For any finite group G and $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ we denote by $E_{(G,\Phi)}$ the subfunctor of $B^A_{\mathbb{K}}$ generated by $e^G_{(G,\Phi)}$. In other words, for each finite group H, one has

$$E_{(G,\Phi)}(H) = \{x \cdot_G e_{(G,\Phi)} \mid x \in B^A_{\mathbb{K}}(H,G)\} \,.$$

7.2 Proposition *For $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$, the following are equivalent:*

(i) *If H is a finite group with $E_{(G,\Phi)}(H) \neq \{0\}$ then $|G| \leqslant |H|$.*

(ii) *If H is a finite group with $E_{(G,\Phi)}(H) \neq \{0\}$ then G is isomorphic to a subquotient of H.*

(iii) *For all $\{1\} \neq N \trianglelefteq G$ one has $m^N_{(G,\Phi)} = 0$.*

(iv) *For all $\{1\} \neq N \trianglelefteq G$ one has $\mathrm{def}^G_{G/N}(e^G_{(G,\Phi)}) = 0$.*

Proof That (ii) implies (i) and that (i) implies (iv) follows from the definitions. Moreover, that (iii) and (iv) are equivalent follows from Proposition 4.3(a). So, it suffices to prove that (iii) implies (ii).

Assume that (iii) holds and let H be a finite group with $E_{(G,\Phi)}(H) \neq \{0\}$. By the definition of $E_{(G,\Phi)}$ this implies that there exists $(U,\phi) \in \mathcal{M}(G\times H)$ such that $\left[\frac{H\times G}{U,\phi}\right] \cdot_G e^G_{(G,\Phi)} \neq 0$. The canonical decomposition of $\left[\frac{H\times G}{U,\phi}\right]$ from Theorem 2.3 implies that

$$\left[\frac{(P/K)\times(Q/L)}{\bar{U},\bar{\phi}}\right] \underset{p_2(U)/\ker(\phi_2)}{\cdot} \mathrm{def}^Q_{Q/L} \underset{Q}{\cdot} \mathrm{res}^G_Q \underset{G}{\cdot} e^G_{(G,\Phi)} \neq 0 \,,$$

with $P := p_1(U)$, $K := \ker(\phi_1)$, $Q := p_2(U)$, $L := \ker(\phi_2)$, $\bar{U}$ corresponding to $U/(K \times L)$ via the canonical isomorphism $(P \times Q)/(K \times L) \cong (P/K) \times (Q/L)$, and $\bar{\phi} \in (\bar{U})^*$ induced by ϕ. Proposition 4.1(c) implies that $G = Q$, and then Proposition 4.3(a) implies that $L = \{1\}$. Thus, $p_1(\bar{U}) = P/K$, $p_2(\bar{U}) = G$, $k_2(\bar{U}) = \{1\}$ and

$$\left[\frac{P/K \times G}{\bar{U}, \bar{\phi}}\right] \underset{G}{\cdot} e^G_{(G,\Phi)} \neq 0 \,.$$

Lemma 5.2 implies that $(\bar{\phi})_2$ extends to G and Lemma 5.1 implies that $\bar{\phi}$ extends to some $\alpha \times \beta \in ((P/K) \times G)^*$ with $\alpha \in (P/K)^*$ and $\beta \in G^*$, since $p_1(\bar{U}) = P/K$ and $p_2(\bar{U}) = G$. Further, we have

$$0 \neq \left[\frac{(P/K) \times G}{\bar{U}, \bar{\phi}}\right] \underset{G}{\cdot} e^G_{(G,\Phi)} = \mathrm{tw}_\alpha \underset{P/K}{\cdot} \left[\frac{(P/K) \times G}{\bar{U}, 1}\right] \underset{G}{\cdot} \mathrm{tw}_\beta \underset{G}{\cdot} e^G_{(G,\Phi)}$$

with $\mathrm{tw}_\beta \underset{G}{\cdot} e^G_{(G,\Phi)} = \Phi(\beta) e^G_{(G,\Phi)}$ by Proposition 4.5(c). Thus, using the canonical decomposition of $\left[\frac{(P/K)\times G}{\bar{U},1}\right]$ as in Theorem 2.3 we obtain $\mathrm{def}^G_{G/k_2(\bar{U})} \cdot_G e^G_{(G,\Phi)} \neq 0$. Proposition 4.3(a) implies that $k_2(\bar{U}) = \{1\}$. But this implies that $G \cong G/\{1\} \cong p_2(\bar{U})/k_2(\bar{U}) \cong p_1(\bar{U})/k_1(\bar{U})$ is isomorphic to a subquotient of $\bar{U}$, which is isomorphic to a subquotient of H. □

Note that by the formula for $m^N_{(G,\Phi)}$ in Proposition 4.3, the condition in Proposition 7.2(iii) is independent of the choice of $\mathbb{K}$ as long as $\mathbb{K}$ has enough roots of unity.

7.3 Definition Let G and H be finite groups and let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and $\Psi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$.

(a) The pair (G, Φ) is called a *B^A-pair* if the equivalent conditions in Proposition 7.2 are satisfied.

(b) We call (G, Φ) and (H, Ψ) *isomorphic* and write $(G, \Phi) \cong (H, \Psi)$ if there exists an isomorphism $f\colon H \xrightarrow{\sim} G$ such that $\Psi \circ f^* = \Phi$. Note that if $(G, \Phi) \cong (H, \Psi)$ then $E_{(G,\Phi)} = E_{(H,\Psi)}$. In fact, if $f\colon H \xrightarrow{\sim} G$ satisfies $\Psi \circ f^* = \Phi$ then $\mathrm{iso}_f(e^H_{(H,\Psi)}) = e^G_{(G,\Phi)}$, by Proposition 4.5(a).

(c) We write $(H, \Psi) \preccurlyeq (G, \Phi)$ if there exists a normal subgroup N of G such that $(H, \Psi) \cong (G/N, \Phi_N)$. The relation $\preccurlyeq$ is reflexive and transitive. Moreover, if $(H, \Psi) \preccurlyeq (G, \Phi)$ and $(G, \Phi) \preccurlyeq (H, \Psi)$ then $(G, \Phi) \cong (H, \Psi)$. Thus, $\preccurlyeq$ induces a partial order on the set of isomorphism classes $[G, \Phi]$ of pairs (G, Φ), where G is a finite group and $\Phi \in \mathrm{Hom}(G, \mathbb{K}^\times)$. We denote this relation again by $\preccurlyeq$. This partial order restricts to a partial order on the set $\mathcal{B}^A$ of isomorphism classes of B^A-pairs.

7.4 Proposition *Let G and H be finite groups and let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and $\Psi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$.*

(a) *If $(H, \Psi) \preccurlyeq (G, \Phi)$ then $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$.*

(b) *If (H, Ψ) is a B^A-pair and $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$ then $(H, \Psi) \preccurlyeq (G, \Phi)$.*

Proof (a) Let $N \trianglelefteq G$ and let $f\colon H \xrightarrow{\sim} G/N$ be an isomorphism with $\Psi \circ f^* = \Phi_N$. Then, by Proposition 4.5(a) and Proposition 4.2(b), we have

$$e^G_{(G,\Phi)} = e^G_{(G,\Phi)} \cdot (\mathrm{inf}^G_{G/N} \underset{G/N}{\cdot} \mathrm{iso}_f \underset{H}{\cdot} e^H_{(H,\Psi)}) \in E_{(H,\Psi)}(G) \,,$$

so that $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$.

(b) Since $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$, we have $e^G_{(G,\Phi)} \in E_{(H,\Psi)}(G)$ and there exists $(U,\phi) \in \mathcal{M}(G\times H)$ such that

$$0 \neq e^G_{(G,\Phi)} \cdot \left(\left[\frac{G\times H}{U,\phi}\right] \cdot_H e^H_{(H,\Psi)} \right) . \tag{29}$$

Lemma 5.3 implies that $p_1(U) = G$, $p_2(U) = H$, ϕ extends to some $\alpha\times\beta \in (G\times H)^*$, $\Phi_{k_1(U)} = \Psi_{k_2(U)} \circ \eta_U^*$, and $m^{k_2(U)}_{(H,\Psi)} \neq 0$. Since (H,Ψ) is a B^A-pair, we obtain $k_2(U) = \{1\}$ and $\Phi_{k_1(U)} = \Psi\circ\eta_U^*$. Thus, η_U is an isomorphism $H \xrightarrow{\sim} G/k_1(U)$ with $\Phi_{k_1(U)} = \Psi \circ \eta_U^*$, so that $(H,\Psi) \preccurlyeq (G,\Phi)$. □

8 Subfunctors of $B^A_{\mathbb{K}}$

We keep the assumptions on $\mathbb{K}$ from Section 7. In this section we prove one of our main results, Theorem 8.8, which describes the lattice of subfunctors of $B^A_{\mathbb{K}}$ in terms of the poset $(\mathcal{B}^A, \preccurlyeq)$.

8.1 Definition Let k be a commutative ring and $F \in \mathcal{F}^G_k$. A finite group G is called a *minimal group* for F if G is a group of minimal order with $F(G) \neq \{0\}$.

For any finite group G, the group $\mathrm{Aut}(G)$ acts on $\mathcal{X}(G)$ via ${}^f(K,\Psi) := (f(K), \Psi\circ(f|_K)^*)$. We will denote by $\hat{\mathcal{X}}(G) \subseteq \mathcal{X}(G)$ the set of those pairs (K,Ψ) with $K = G$. Note that $\hat{\mathcal{X}}(G)$ is $\mathrm{Aut}(G)$-invariant and that G acts trivially by conjugation on $\hat{\mathcal{X}}(G)$, so that $\hat{\mathcal{X}}(G)$ can be viewed as an $\mathrm{Out}(G)$-set.

8.2 Proposition *Let F be a subfunctor of $B^A_{\mathbb{K}}$ in $\mathcal{F}^A_{\mathbb{K}}$.*

(a) *For each finite group G one has*

$$F(G) = \bigoplus_{(K,\Psi)\in[G\backslash\mathcal{X}_F(G)]} \mathbb{K}\cdot e^G_{(K,\Psi)}\,,$$

where $\mathcal{X}_F(G) := \{(K,\Psi)\in\mathcal{X}(G) \mid e^G_{(K,\Psi)} \in F(G)\}$.

(b) *For any finite group G, the set $\mathcal{X}_F(G)$ is invariant under the action of $\mathrm{Aut}(G)$.*

(c) *Suppose that H is a minimal group for F, i.e., of minimal order with $F(H)\neq\{0\}$. Then $\mathcal{X}_F(H)$ contains only elements of the form (K,Ψ) with $K = H$. Moreover, each $(H,\Psi)\in\mathcal{X}_F(H)$ is a B^A-pair and one has $E_{(H,\Psi)} \subseteq F$.*

Proof (a) For all $a\in F(G)$ and $x \in B^A_{\mathbb{K}}(G)$, [CY19, Proposition 2] implies $x\cdot a = \Delta(x)\cdot_G a \in F(G)$. Thus, $F(G)$ is an ideal of $B^A_{\mathbb{K}}(G)$. Since the elements $e^G_{(K,\Theta)}$ with $(K,\Theta)\in[G\backslash\mathcal{X}(G)]$ form a $\mathbb{K}$-basis of $B^A_{\mathbb{K}}(G)$ consisting of pairwise orthogonal idempotents, the assertion in (a) follows.

(b) If $e^G_{(K,\Psi)} \in F(G)$ and $f\in\mathrm{Aut}(G)$ then $e^G_{(f(K),\Psi\circ(f|_K)^*)} = \mathrm{iso}_f(e^G_{(K,\Psi)}) \in F(G)$.

(c) Assume that H is a minimal group for F and that $(K,\Psi)\in\mathcal{X}_F(H)$. Then $e^H_{(K,\Psi)}\in F(H)$ and

$$0 \neq e^K_{(K,\Psi)} = e^K_{(K,\Psi)}\mathrm{res}^H_K(e^H_{(K,\Psi)}) \in F(K)\,,$$

by Proposition 4.1(b). The minimality of H implies $K = H$. By Proposition 4.3(a), the minimality also implies that $m^N_{(H,\Psi)} = 0$ for all $\{1\}\neq N \trianglelefteq H$, since $e^H_{(H,\Psi)}\in F(H)$. Clearly, $E_{(H,\Psi)}\subseteq F$. □

8.3 Definition Let F be a subfunctor of $B_{\mathbb{K}}^A$ in $\mathcal{F}_{\mathbb{K}}^A$. If H is a minimal group for F and $\Psi \in \mathrm{Hom}(H, \mathbb{K}^\times)$ is such that $(H, \Psi) \in \mathcal{X}_F(H)$ then we call (H, Ψ) a *minimal pair* for F. By Proposition 8.2(c), each minimal pair for F is a B^A-pair.

8.4 Proposition *Let H be a finite group, $\Psi \in \mathrm{Hom}(H^*, \mathbb{K}^*)$, and let (G, Φ) be a minimal pair for $E_{(H,\Psi)}$. Then:*

(a) $E_{(H,\Psi)} = E_{(G,\Phi)}$.

(b) *There exists $N \trianglelefteq H$ with $m_{(H,\Psi)}^N \neq 0$ and $(H/N, \Psi_N) \cong (G, \Phi)$. In particular $(G, \Phi) \preccurlyeq (H, \Psi)$. Moreover, if also $N' \trianglelefteq H$ satisfies $(H/N', \Psi_{N'}) \cong (G, \Phi)$ then $m_{(H,\Psi)}^{N'} = m_{(H,\Psi)}^N \neq 0$.*

(c) *Up to isomorphism, (G, Φ) is the only minimal pair for $E_{(H,\Psi)}$.*

(d) *If (H, Ψ) is a B^A-pair, then, up to isomorphism, (H, Ψ) is the only minimal pair of $E_{(H,\Psi)}$. In particular,*

$$E_{(H,\Psi)}(H) = \bigoplus_{\substack{(H,\Psi') \in \hat{\mathcal{X}}(H) \\ (H,\Psi') =_{\mathrm{Out}(H)} (H,\Psi)}} \mathbb{K} e_{(H,\Psi')}^H .$$

Proof (b) Since (G, Φ) is a minimal pair for $E_{(H,\Psi)}$, there exists $x \in B_{\mathbb{K}}^A(G, H)$ such that $e_{(G,\Phi)}^G = x \cdot_H e_{(H,\Psi)}$. Multiplication with $e_{(G,\Phi)}^G$ yields $e_{(G,\Phi)}^G = e_{(G,\Phi)}^G \cdot (x \cdot_H e_{(H,\Psi)}^H)$. Thus, there exists $(U, \phi) \in \mathcal{M}(G \times H)$ with

$$e_{(G,\Phi)}^G \cdot \left(\left[\frac{G \times H}{U, \phi} \right] \underset{H}{\cdot} e_{(H,\Psi)}^H \right) \neq 0 .$$

Lemma 5.3 implies that $p_1(U) = G$, ϕ has an extension to $G \times H$, $\Phi_{k_1(U)} = \Psi_{k_2(U)} \circ \eta_U^*$, and $m_{(H,\Psi)}^{k_2(U)} \neq 0$. Since ϕ has an extension to $G \times H$, $\left[\frac{G \times H}{U, \phi}\right]$ factors through $q(U) \cong G/k_1(U)$ by Lemma 5.1. Since G is a minimal group of $E_{(H,\Psi)}$ this implies $k_1(U) = \{1\}$. Set $N := k_2(U) \trianglelefteq H$. Then $\eta_U \colon H/N \xrightarrow{\sim} G$ satisfies $\Psi_N \circ \eta_U^* = \Phi$. If also $N' \trianglelefteq G$ satisfies $(H/N', \Psi_{N'}) \cong (G, \Phi)$, then $(H/N, \Psi_{N'}) \cong (H/N, \Psi_N)$ and we obtain $m_{(H,\Psi)}^{N'} = m_{(H,\Psi)}^N \neq 0$ by Proposition 6.4.

(a) Note that $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$ by Proposition 8.2(c). The converse follows from Proposition 7.4(a) and Part (b).

(c) Assume that also (G', Φ') is a minimal pair for $E_{(H,\Phi)} = E_{(G,\Phi)}$. Then, by Part (b) applied to (G', Φ') and (G, Φ) in place of (G, Φ) and (H, Ψ), we have $(G', \Phi') \preccurlyeq (G, \Phi)$. Since both G and G' are minimal groups for $E_{(H,\Psi)}$, they have the same order. Thus, $(G', \Phi') \cong (G, \Phi)$.

(d) Now assume that (H, Ψ) is a B^A-pair and let (G, Φ) be a minimal pair for $E_{(H,\Psi)}$. Then, by Part (b), there exists $N \trianglelefteq G$ with $m_{(H,\Psi)}^N \neq 0$ and $(H/N, \Psi_N) \cong (G, \Phi)$. Since (H, Ψ) is a B^A-pair, this implies $N = \{1\}$ and $(H, \Psi) \cong (G, \Phi)$. □

8.5 Notation For any finite group G and any $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ we denote by $\beta(G, \Phi)$ the class of all minimal pairs for $E_{(G,\Phi)}$. Thus $\beta(G, \Phi)$ is the isomorphism class $[H, \Psi]$ of a B^A-pair (H, Ψ). Note that $\beta(G, \Phi) \preccurlyeq [G, \Phi]$ by Proposition 8.4(b).

The following proposition is not used in this paper, but of interest in its own right. It is the analogue of [Bc10, Theorem 5.4.11].

8.6 Proposition *Let G be a finite group and let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$.*

(a) *If (H, Ψ) is a B^A-pair with $(H, \Psi) \preccurlyeq (G, \Phi)$ then $[H, \Psi] \preccurlyeq \beta(G, \Phi)$.*

(b) *For any $N \trianglelefteq G$ the following are equivalent:*

(i) $m^N_{(G,\Phi)} \neq 0$.

(ii) $\beta(G, \Phi) \preccurlyeq [G/N, \Phi_N]$.

(iii) $\beta(G, \Phi) = \beta(G/N, \Phi_N)$.

(c) *For any $N \trianglelefteq G$ the following are equivalent:*

(i) $[G/N, \Phi_N] \cong \beta(G, \Phi)$.

(ii) $(G/N, \Phi_N)$ *is a B^A-pair and* $m^N_{(G,\Phi)} \neq 0$.

Proof Let $(K, \Theta) \in \beta(G, \Phi)$. Thus, $E_{(G,\Phi)} = E_{(K,\Theta)}$ by Proposition 8.4(a) and (K, Θ) is a B^A-pair.

(a) Let (H, Ψ) be as in the statement. Then $E_{(K,\Theta)} = E_{(G,\Phi)} \subseteq E_{(H,\Phi)}$ by Proposition 7.4(a). Now Proposition 7.4(b) implies $(H, \Psi) \preccurlyeq (K, \Theta)$.

(b) (i)⇒(ii): Since $m^N_{(G,\Phi)} \neq 0$, Proposition 4.3(a) implies that

$$e^{G/N}_{(G/N,\Phi_N)} = (m^N_{(G,\Phi)})^{-1} \mathrm{def}^G_{G/N}(e^G_{(G,\Phi)}) \in E_{(G,\Phi)}(G/N) = E_{(K,\Theta)}(G/N)$$

so that $E_{(G/N,\Phi_N)} \subseteq E_{(K,\Theta)}$. Proposition 7.4(b) implies $(K, \Psi) \preccurlyeq (G/N, \Phi_N)$.

(ii)⇒(iii): By Part (a) applied to (K, Θ) and $(G/N, \Phi_N)$ we obtain $\beta(G, \Phi) = [K, \Theta] \preccurlyeq \beta(G/N, \Phi_N)$. Conversely, we have $\beta(G/N, \Phi_N) \preccurlyeq [G/N, \Phi_N] \preccurlyeq [G, \Phi]$ and Part (a) again implies $\beta(G/N, \Phi_N) \preccurlyeq \beta(G, \Phi)$.

(iii)⇒(i): By Proposition 8.4(b) there exists $M \trianglelefteq G$ such that $m^M_{(G,\Phi)} \neq 0$ and $[G/M, \Phi_M] \cong \beta(G, \Phi)$. Similarly, there exists $N \leqslant M' \trianglelefteq G$ such that $m^{M'/N}_{(G/N,\Phi_N)} \neq 0$ and $[(G/N)/(M'/N), (\Phi_N)_{M'}] = \beta(G/N, \Phi_N)$. Since

$$[G/M', \Phi_{M'}] = [(G/N)/(M'/N), (\Phi_N)_{M'}] = \beta(G/N, \Phi_N) = \beta(G, \Phi) = [G/M, \Phi_M] ,$$

Proposition 6.4 implies that $m^{M'}_{(G,\Phi)} = m^M_{(G,\Phi)} \neq 0$. By Proposition 6.1 we have $m^{M'}_{(G,\Phi)} = m^N_{(G,\Phi)} \cdot m^{M'}_{(G/N,\Phi_N)}$ which implies that $m^N_{(G,\Phi)} \neq 0$.

(c) This follows immediately from the equivalence between (i) and (iii) in Part (b), noting that $\beta(G/N, \Phi_N) = (G/N, \Phi_N)$ if $(G/N, \Phi_N)$ is a B^A-pair and that $\beta(G, \Phi)$ consists of B^A-pairs. □

8.7 Definition A subset $\mathcal{Z}$ of the poset $\mathcal{B}^A$, ordered by the relation $\preccurlyeq$ (cf. Definition 7.3), is called *closed* if for every $[H, \Psi] \in \mathcal{Z}$ and $[G, \Phi] \in \mathcal{B}^A$ with $[H, \Psi] \preccurlyeq [G, \Phi]$ one has $[G, \Phi] \in \mathcal{Z}$.

8.8 Theorem *Let $\mathcal{S}$ denote the set of subfunctors of $B^A_{\mathbb{K}}$ in $\mathcal{F}^A_{\mathbb{K}}$, ordered by inclusion of subfunctors, and let $\mathcal{T}$ denote the set of closed subsets of $\mathcal{B}^A$, ordered by inclusion of subsets. The map*

$$\alpha\colon \mathcal{S} \to \mathcal{T}\,, \quad F \mapsto \{[H, \Psi] \in \mathcal{B}^A \mid E_{(H,\Psi)} \subseteq F\}$$

is an isomorphism of posets with inverse given by

$$\beta\colon \mathcal{T} \to \mathcal{S}\,, \quad \mathcal{Z} \mapsto \sum_{[H,\Psi]\in\mathcal{Z}} E_{(H,\Psi)}\,.$$

Proof Clearly, α and β are order-preserving. Let $F \in \mathcal{S}$. By Proposition 8.2(a) we have

$$F = \sum_{\substack{G \\ (H,\Psi)\in\mathcal{X}_F(G)}} \langle e^G_{(H,\Psi)} \rangle \,,$$

where G runs through a set of representatives of the isomorphism classes of finite groups and $\langle e^G_{(H,\Psi)} \rangle$ denotes the subfunctor of $B^A_{\mathbb{K}}$ generated by $e^G_{(H,\Psi)}$. For any finite group G and any $(H,\Psi) \in \mathcal{X}(G)$ one has $e^G_{(H,\Psi)} \in F(G)$ if and only if $e^H_{(H,\Psi)} \in F(H)$. In fact, $e^H_{(H,\Psi)} = e_{(H,\Psi)} \cdot \mathrm{res}^G_H(e^G_{(H,\Psi)})$ by Proposition 4.1 and $e^G_{(H,\Psi)} \in \mathbb{K} \cdot \mathrm{ind}^G_H(e^H_{(H,\Psi)})$ by Proposition 4.4. Thus

$$F = \sum_{\substack{H \\ (H,\Psi)\in\hat{\mathcal{X}}_F(H)}} E_{(H,\Psi)} \,,$$

where H runs again through a set of representatives of the isomorphism classes of finite groups and $\hat{\mathcal{X}}_F(H) = \hat{\mathcal{X}}(H) \cap \mathcal{X}_F(H)$. By Propositions 8.2(c) and 8.4(a), we obtain

$$F = \sum_{\substack{[H,\Psi]\in\mathcal{B}^A \\ (H,\Psi)\in\mathcal{X}_F(H)}} E_{(H,\Psi)} = \sum_{[H,\Psi]\in\alpha(F)} E_{(H,\Psi)} = \beta(\alpha(F)) \,,$$

since $(H,\Psi) \in \mathcal{X}_F(H)$ if and only if $E_{(H,\Psi)} \subseteq F$.

Let $\mathcal{Z}$ be a closed subset of $\mathcal{B}^A$. By definition of α and β we have

$$\alpha(\beta(\mathcal{Z})) = \{[H,\Psi] \in \mathcal{B}^A \;\mid E_{(H,\Psi)} \subseteq \sum_{[G,\Phi]\in\mathcal{Z}} E_{(G,\Phi)}\} \,.$$

The inclusion $\mathcal{Z} \subseteq \alpha(\beta(\mathcal{Z}))$ is obvious. Conversely, assume that $[H,\Psi] \in \mathcal{B}^A$ satisfies $E_{(H,\Psi)} \subseteq \sum_{[G,\Phi]\in\mathcal{Z}} E_{(G,\Phi)}$. Evaluation at H and Proposition 8.2(a) imply that there exists $[G,\Phi] \in \mathcal{Z}$ with $e^H_{(H,\Psi)} \in E_{(G,\Phi)}(H)$, which implies $E_{(H,\Psi)} \subseteq E_{(G,\Phi)}$. Since (G,Φ) is a B^A-pair, Proposition 7.4(b) implies $[G,\Phi] \preccurlyeq [H,\Psi]$. Since $[G,\Phi] \in \mathcal{Z}$ and $\mathcal{Z}$ is closed we obtain $[H,\Psi] \in \mathcal{Z}$. Thus, $\alpha(\beta(\mathcal{Z})) \subseteq \mathcal{Z}$, and the proof is complete. □

8.9 Remark (a) If (G,Φ) is a B^A-pair, then the subfunctor $E_{(G,\Phi)}$ of $B^A_{\mathbb{K}}$ corresponds under the bijection in Theorem 8.8 to the subset $\mathcal{B}^A_{\succcurlyeq[G,\Phi]} := \{[H,\Psi] \in \mathcal{B}^A \mid [G,\Phi] \preccurlyeq [H,\Psi]\}$. Clearly, $\mathcal{B}^A_{\succ[G,\Phi]} := \{[H,\Psi] \in \mathcal{B}^A \mid [G,\Phi] \prec [H,\Psi]\}$ is the unique maximal closed subset of $\mathcal{B}^A_{\succcurlyeq[G,\Phi]}$.

(b) For every element $[G,\Phi] \in \mathcal{B}^A$ there exist only finitely many elements $[H,\Psi] \in \mathcal{B}^A$ with $[H,\Psi] \preccurlyeq [G,\Phi]$. Therefore, every non-empty subset of $\mathcal{B}^A$ has a minimal element.

9 Composition factors of $B^A_{\mathbb{K}}$

We keep the assumptions on $\mathbb{K}$ from Section 7. In this section we show that the composition factors of $B^A_{\mathbb{K}}$ are parametrized by isomorphism classes of B^A-pairs.

9.1 Recall from [BC18, Section 9] that the simple A-fibered biset functors S over $\mathbb{K}$ are parametrized by isomorphism classes of certain quadruples (G, K, κ, V). Here, G is a minimal group for S, $(K, \kappa) \in \mathcal{M}(G)$ is such that the idempotent $f_{(K,\kappa)} \in B_{\mathbb{K}}^A(G, G)$ (see [BC18, Subsection 4.3]) does not annihilate $S(G)$, and $V := S(G)$ is an irreducible $\mathbb{K}\Gamma_{(G,K,\kappa)}$-module for the finite group $\Gamma_{(G,K,\kappa)}$ defined in [BC18, 6.1(c)]. All we need for the analysis in the following theorem is that the idempotent $f_{(K,\kappa)}$ is in the $\mathbb{K}$-span of standard basis elements $\left[\frac{G\times G}{U,\phi}\right]$ with $(U, \phi) \in \mathcal{M}(G, G)$ such that $k_2(U) \geqslant K$, and that in the case $(K, \kappa) = (\{1\}, 1)$, the group $\Gamma_{(G,\{1\},1)}$ is the set of standard basis elements $\left[\frac{G\times G}{U,\phi}\right]$ of $B_{\mathbb{K}}^A(G, G)$ with $p_1(U) = G = p_2(U)$ and $k_1(U) = \{1\} = k_2(U)$. The multiplication is given by $\cdot_G$. Thus, in this case, $\left[\frac{G\times G}{U,\phi}\right] = \mathrm{tw}_\alpha \cdot_G \mathrm{iso}_f$, where $f := \eta_U \in \mathrm{Aut}(G)$ is defined by $U = \{(\eta_U(g), g) \mid g \in G\}$, and $\alpha \in G^*$ is given by $\alpha(g) = \phi(\eta_U(g), g)$ for $g \in G$. Mapping $\left[\frac{G\times G}{U,\phi}\right]$ to the element $(\alpha, \bar{f}) \in G^* \rtimes \mathrm{Out}(G)$ defines an isomorphism. Here, $\mathrm{Out}(G)$ acts on G^* via ${}^{\bar{f}}\alpha := \alpha \circ f^*$. Moreover, $\Gamma_{(G,\{1\},1)}$ acts on $S(G)$ by $\cdot_G$.

9.2 Proposition *Let (G, Φ) be a B^A-pair. The subfunctor $E_{(G,\Phi)}$ of $B_{\mathbb{K}}^A$ has a unique maximal subfunctor $J_{(G,\Phi)}$, given by*

$$J_{(G,\Phi)} = \sum_{[H,\Psi]\in\mathcal{B}^A_{\succ[G,\Psi]}} E_{(H,\Psi)}\,.$$

The simple functor $S_{(G,\Phi)} := E_{(G,\Phi)}/J_{(G,\Phi)}$ is isomorphic to $S_{(G,\{1\},1,V_\Phi)}$ where V_Φ is the irreducible $\mathbb{K}[G^* \rtimes \mathrm{Out}(G)]$-module

$$V_\Phi := \mathrm{Ind}_{G^*\rtimes\mathrm{Out}(G)_\Phi}^{G^*\rtimes\mathrm{Out}(G)}(\mathbb{K}_{\tilde{\Phi}})\,,$$

with $\tilde{\Phi} \in \mathrm{Hom}(G^* \rtimes \mathrm{Out}(G)_\Phi, \mathbb{K}^\times)$ defined by $\tilde{\Phi}(\phi, \bar{f}) := \Phi(\phi)$ for $\phi \in G^*$ and $f \in \mathrm{Aut}(G)$.

Proof By Remark 8.9(a) and Theorem 8.8, $J_{(G,\Phi)}$ is the unique maximal subfunctor of $E_{(G,\Phi)}$. Thus, the functor $S := S_{(G,\Phi)}$ is a simple object in $\mathcal{F}_{\mathbb{K}}^A$. Moreover, G is a minimal group for S, since G is a minimal group for $E_{(G,\Phi)}$ and $E_{(H,\Psi)}(G) = \{0\}$ for all $[H, \Psi] \in \mathcal{B}^A_{\succ[G,\Phi]}$.

Let $(U, \phi) \in \mathcal{M}(G \times G)$ with $k_2(U) \neq \{1\}$. Then $\left[\frac{G\times G}{U,\phi}\right]$ factors through the group $q(U)$ which has smaller order than G. Thus, $\left[\frac{G\times G}{U,\phi}\right] \cdot_G e^G_{(G,\Phi')} = 0$ for all $(G, \Phi') \in \hat{\mathcal{X}}(G)$ with $(G', \Phi') =_{\mathrm{Out}(G)} (G, \Phi)$. By Proposition 8.4(d) this yields $\left[\frac{G\times G}{U,\phi}\right] \cdot_G S(G) = \{0\}$. Thus, $f_{(K,\kappa)} \cdot_G S(G) = 0$ for all (K, κ) with $|K| > 1$.

This implies that S is parametrized by the quadruple $(G, \{1\}, 1, V)$, with $V = S(G)$ viewed as $\mathbb{K}\Gamma_{(G,\{1\},1)}$-module. Since $S(G)$ is the $\mathbb{K}$-span of the idempotents $e^G_{(G,\Phi')}$, with (G, Φ') running through the $\mathrm{Out}(G)$-orbit of (G, Φ), and since $\mathrm{tw}_\alpha \cdot_G \mathrm{iso}_f \cdot_G e^G_{(G,\Phi')} = \Phi(\alpha) \cdot e^G_{(G,\Phi'\circ f^*)}$ for all $\alpha \in G^*$ and $f \in \mathrm{Aut}(G)$ and (G, Φ'), the $\mathbb{K}\Gamma_{(G,\{1\},1)}$-module $S(G)$ is monomial. The stabilizer of the one-dimensional subspace $\mathbb{K}e^G_{(G,\Phi)}$ is equal to $G^* \rtimes \mathrm{Out}(G)_\Phi$ and this group acts on $\mathbb{K}e^G_{(G,\Phi)}$ via $\tilde{\Phi}$. Thus, $S(G) \cong V_\Phi$ as $\mathbb{K}\Gamma_{(G,\{1\},1)}$-module and the proof is complete. □

9.3 Theorem *Let $F' \subset F \subseteq B_{\mathbb{K}}^A$ be subfunctors in $\mathcal{F}_{\mathbb{K}}^A$ such that F/F' is simple. Then there exists a unique $[G, \Phi] \in \mathcal{B}^A$ such that $E_{(G,\Phi)} \subseteq F$ and $E_{(G,\Phi)} \not\subseteq F'$. Moreover, $E_{(G,\Phi)} + F' = F$ and $E_{(G,\Phi)} \cap F' = J_{(G,\Phi)}$, and $F/F' \cong S_{(G,\Phi)}$.*

Proof Since $\alpha(F')$ is a maximal subset of $\alpha(F)$ and both are closed, it follows from Theorem 8.8 and Remark 8.9 that $\alpha(F) \smallsetminus \alpha(F') = \{[G,\Phi]\}$ for a unique $[G,\Phi] \in \mathcal{B}^A$. For any $[H,\Psi] \in \mathcal{B}^A$ one has $E_{(H,\Psi)} \subseteq F$ and $E_{(H,\Psi)} \not\subseteq F'$ if and only if $[H,\Psi] \in \alpha(F)$ but $[H,\Psi] \notin \alpha(F')$. Thus, the first condition is equivalent to $[H,\Psi] = [G,\Phi]$. Further, we have $F' \subset F' + E_{(G,\Phi)} \subseteq F$ which implies $F' + E_{(G,\Phi)} = F$, since F/F' is simple. Thus, $0 \neq E_{(G,\Phi)}/(E_{(G,\Phi)} \cap F') \cong (E_{(G,\Phi)} + F')/F' = F/F'$, and by Proposition 9.2 we obtain $E_{(G,\Phi)} \cap F' = J_{(G,\Phi)}$ so that $F/F' \cong E_{(G,\Phi)}/J_{(G,\Phi)} \cong S_{(G,\Phi)}$. □

10 The case $A \leqslant \mathbb{K}^\times$

In this section we assume that A is a subgroup of the unit group of a field $\mathbb{K}$ of characteristic 0. Then the assumptions on A and $\mathbb{K}$ from the beginnings of Sections 3–9 are satisfied. This special case has been used for instance in the theory of canonical induction formulas, see [Bo98]. This assumption was also used in [Ba04] and [CY19]. By double duality it allows to consider pairs $(G, gO(G))$ for a normal subgroup $O(G)$ of G instead of pairs (G,Φ) with $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$. This section makes this translation precise and also translates previously defined features for pairs (G,Φ) to features for pairs $(G, gO(G))$.

For any finite group G we have a homomorphism

$$\zeta_G \colon G \to \mathrm{Hom}(G^*, \mathbb{K}^\times)\,, \quad g \mapsto \varepsilon_g\,, \quad \text{with } \varepsilon_g(\phi) := \phi(g),$$

for $\phi \in G^*$. Note that ζ_G is functorial in G, i.e., if $f \colon G \to H$ is a group homomorphism then $\zeta_H \circ f = \mathrm{Hom}(f^*, \mathbb{K}^\times) \circ \zeta_G$. We set

$$O^A(G) := O(G) := \ker(\zeta_G) = \bigcap_{\phi \in G^*} \ker(\phi)\,,$$

which is a normal subgroup of G containing the commutator subgroup $[G,G]$ of G. Thus, we obtain an injective homomorphism $\bar{\zeta}_G \colon G/O(G) \to \mathrm{Hom}(G^*, \mathbb{K}^\times)$.

10.1 Proposition *Let G be a finite group.*

(a) *The homomorphism ζ_G is surjective and $\bar{\zeta}_G \colon G/O(G) \xrightarrow{\sim} \mathrm{Hom}(G^*, \mathbb{K}^\times)$ is an isomorphism.*

(b) *The subgroup $O(G)$ is the smallest subgroup $[G,G] \leqslant M \leqslant G$ such that A has an element of order $\exp(G/M)$.*

(c) *For any normal subgroup N of G one has $O(G/N) = O(G)N/N$.*

Proof (a) Applying the functoriality with respect to the natural epimorphism $f \colon G \to G/[G,G]$, and using that f^* is an isomorphism, it suffices to show the statement when G is abelian. Since $\mathrm{Hom}(-^*, \mathbb{K}^\times)$ preserves direct products of abelian groups, we are reduced to the case that G is cyclic. Using again the functoriality with respect to the natural epimorphism onto the largest quotient of G whose order occurs as an element order in A, we are reduced to the case that G is cyclic of order n and A has an element of order n. In this case it is easy to see that ζ_G is injective and that G and $\mathrm{Hom}(G^*, \mathbb{K}^\times)$ have the same order.

(b) First note that if M_1 and M_2 have the stated property, then also $M_1 \cap M_2$ has this property. In fact, $G/(M_1 \cap M_2)$ is isomorphic to a subgroup of $G/M_1 \times G/M_2$, whose exponent is equal to the order of an element in A. Here we use that if elements a and b in A have orders k and l

respectively, then A has an element whose order is the least common multiple of k and l. Thus, there exists a smallest subgroup M with the stated property. Clearly, $\ker(\phi)$ has the property for every $\phi \in G^*$. Therefore, also $O(G)$ has the desired property. Conversely, if M has the property, then by writing G/M as a direct product of n cyclic groups whose orders are achieved as element order in A, it is easy to construct elements $\phi_1, \ldots, \phi_n \in G^*$ such that $\bigcap_{i=1}^n \ker(\phi_i) = M$, implying that $O(G) \leqslant M$.

(c) Since the exponent of $G/O(G)$ is equal to the order of an element of A also the exponent of $(G/N)/(O(G)N/N) \cong G/(O(G)N)$ is equal to the order of an element of A. Thus, $O(G/N) \leqslant O(G)N/N$. Conversely,

$$O(G) = \bigcap_{\phi \in G^*} \ker(\phi) \leqslant \bigcap_{\substack{\phi \in G^* \\ \phi|_N = 1}} \ker(\phi)\,,$$

and taking images in G/N yields the reverse inclusion. □

For any finite group G, Proposition 10.1(a) yields a bijection between the set of pairs of the form (G, Φ), with $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and the set of pairs $(G, gO(G))$ with $gO(G) \in G/O(G)$. More precisely, we identify $(G, gO(G))$ with (G, ε_g). The following proposition translates various relevant features of pairs (G, Φ) to features of the corresponding pairs $(G, gO(G))$. The proofs are straightforward and left to the reader.

10.2 Proposition *Let G and H be finite groups.*

(a) *Let $g \in G$ and $h \in H$. Then $(G, gO(G)) \cong (H, hO(H))$ if and only if there exists an isomorphism $f\colon G \to H$ such that $f(g)O(H) = hO(H)$.*

(b) *Let N be a normal subgroup of G, let $g \in G$, and set $\Phi := \varepsilon_g$. Then $\Phi_N = \varepsilon_{gN}$.*

(c) *Let $g \in G$ and $h \in H$. Then $(H, \varepsilon_h) \preccurlyeq (G, \varepsilon_g)$ if and only if there exists a normal subgroup N of G and an isomorphism $f\colon H \xrightarrow{\sim} G/N$ with $f(h) \in gO(G)N$.*

(d) *Let $K \leqslant G$ and $g \in G$. Then $\varepsilon_g|_{K^\perp} = 1$ if and only if $g \in KO(G)$.*

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