Fusion systems of blocks of finite groups over arbitrary fields*

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Abstract

To any block idempotent b of a group algebra kG of a finite group G over a field k of characteristic $p > 0$, Puig associated a fusion system and proved that it is saturated if the k-algebra $kC_G(P)e$ is split, where (P, e) is a maximal kGb-Brauer pair. We investigate in the non-split case how far the fusion system is from being saturated by describing it in an explicit way as being generated by the fusion system of a related block idempotent over a larger field together with a single automorphism of the defect group.

1 Introduction

Let k be a field of characteristic p, let G be a finite group and let b be a block idempotent of kG. Puig defined a fusion system $\mathcal{F}_{(P,e)}(kGb)$ associated to kGb after choosing a maximal kGb-Brauer pair (P, e). Up to category isomorphism, this fusion system does not depend on the choice of (P, e). Puig also proved that $\mathcal{F}_{(P,e)}(kGb)$ is saturated if the k-algebra $kC_G(P)e$ is split. It is known that in the non-split case it can happen that the fusion system associated to kGb is not saturated. In fact, the Sylow axiom can fail, while the extension axiom always holds. In the Main Theorem 5.2 of this paper we establish a precise connection between the fusion systems of related blocks in a Galois extension L/K of fields of characteristic p with Galois group Γ. More precisely, let b be a block idempotent of LG and $\tilde{b}$ the unique block idempotent of KG with $b\tilde{b} = b$. Moreover, let (P, e) be a maximal LGb-Brauer pair and let $\tilde{e}$ be the unique block idempotent of $KC_G(P)$ with $e\tilde{e} = e$. Then $(P, \tilde{e})$ is a maximal $KG\tilde{b}$-Brauer pair and one has an inclusion of the fusion systems

$$\mathcal{F} := \mathcal{F}_{(P,e)}(LGb) \subseteq \mathcal{F}_{(P,\tilde{e})}(KG\tilde{b}) =: \tilde{\mathcal{F}}\,.$$

Theorem 5.2 states that there exists an element $\sigma \in \mathrm{Aut}_{\tilde{\mathcal{F}}}(P)$ such that $\tilde{\mathcal{F}} = \langle \mathcal{F}, \sigma \rangle$. As consequences of the nature of σ we obtain that $\tilde{\mathcal{F}}$ is saturated if and only if $\mathcal{F}$ is saturated and

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p does not divide the index $[\Gamma_b : \Gamma_e] = [K(e) : K(b)]$ of the stabilizers of b and e under the Galois action, or equivalently the degree of the field extensions after adjoining the coefficients of e and b to K. In the case that L is chosen such that $LC_G(P)e$ is split, this gives a particularly handy criterion for a fusion system of a block $KG\tilde{b}$ in the non-split case to be saturated, see Theorem 6.3. The main result allows an alternative easy proof for the known fact that the extension axiom holds also in the non-split case, see Theorem 6.2. Finally, the Main Theorem implies that a weak form of Alperin's fusion theorem holds also for arbitrary block fusion systems, see Theorem 6.5.

1.1 Notation We will use the following standard notations without further notice:

For a group G and $x \in G$, we denote by $c_x\colon G \to G$ the conjugation map $g \mapsto xgx^{-1}$. If k is a commutative ring, its k-linear extension to the group algebra kG is again denoted by $c_x\colon kG \to kG$. We frequently will use left-exponential notation ${}^{x}(-) := c_x$ for these maps. The maps c_x, $x \in G$, define an action of G on kG via k-algebra homomorphisms.

For $H \leqslant G$, we denote by $[G/H]$ a set of representatives of the cosets G/H.

If a group G acts on a set X, we usually denote the stabilizer of an element $x \in X$ by G_x. Moreover, for $H \leqslant G$, we denote by X^H the set of H-fixed points of X.

2 Brauer pairs

Throughout this section, G denotes a finite group, k denotes a field of characteristic $p > 0$, and b denotes a block idempotent of kG, i.e., a primitive idempotent of $Z(kG)$. We recall the definition and properties of Brauer pairs for kG following the treatment in [AKO11, IV.2]. We note that the blanket assumption in [AKO11, IV.2] that k is algebraically closed is not used in the proofs of any of the statements that we cite from there. Alternatively, see also [L18, Sections 5.9 and 6.3].

Recall that, for a p-subgroup P of G, the *Brauer homomorphism* with respect to P is the k-linear projection map $\mathrm{Br}_P\colon (kG)^P \to kC_G(P)$, $\sum_{g\in G} \alpha_g g \mapsto \sum_{g\in C_G(P)} \alpha_g g$. This is a surjective k-algebra homomorphism which respects G-conjugation: $c_x \circ \mathrm{Br}_P = \mathrm{Br}_{{}^xP} \circ c_x\colon (kG)^P \to kC_G({}^xP)$ for $x \in G$. Thus, $\mathrm{Br}_P(b)$ is an idempotent of $Z(kC_G(P)) = (kC_G(P))^{C_G(P)}$. Recall further that a *kG-Brauer pair* is a pair (P, e) consisting of a p-subgroup P of G and a block idempotent e of $kC_G(P)$. If e occurs in the unique decomposition of $\mathrm{Br}_P(b)$ into a sum of primitive idempotents of $Z(kC_G(P))$ (that is, if $\mathrm{Br}_P(b)e = e$), then we call (P, e) a *(kG, b)-Brauer pair*. We denote by $\mathcal{BP}(kG)$ the set of kG-Brauer pairs and by $\mathcal{BP}(kG, b)$ the set of (kG, b)-Brauer pairs. Clearly, $\mathcal{BP}(kG)$ is the disjoint union of the subsets $\mathcal{BP}(kG, b)$, where b runs through the block idempotents of kG. The set $\mathcal{BP}(kG)$ is a G-set under the conjugation action given by ${}^{x}(P, e) := ({}^{x}P, {}^{x}e)$, and the subset $\mathcal{BP}(kG, b)$ is G-stable. Finally, we say that an idempotent i of $(kG)^P$ is *associated* to a kG-Brauer pair (P, e) if

$$e\mathrm{Br}_P(i) = \mathrm{Br}_P(i) \neq 0\,.$$

Note that if i is primitive in $(kG)^P$ then $e\mathrm{Br}_P(i) \neq 0$ implies that $\mathrm{Br}_P(i) \neq 0$ and that $\mathrm{Br}_P(i)$ is primitive in $kC_G(P)$. Thus, $e\mathrm{Br}_P(i) = \mathrm{Br}_P(i)$. One writes $(Q, f) \leqslant (P, e)$ if $Q \leqslant P$ and if any primitive idempotent i of $(kG)^P$ which is associated to (P, e) is also associated to (Q, f), see [AKO11, Definition 2.9]. This relation has the following properties.

2.1 Theorem *([AKO11, Theorems 2.10, 2.16])* (a) *Let $(P, e) \in \mathcal{BP}(kG)$ and let $Q \leqslant P$. Then there exists a unique block idempotent f of $kC_G(Q)$ such that $(Q, f) \leqslant (P, e)$.*

(b) *Let $(Q, f) \leqslant (P, e)$ be in $\mathcal{BP}(kG)$ with $Q \trianglelefteq P$. Then f is the unique block idempotent of $kC_G(Q)$ which is P-stable and satisfies $Br_P(f)e = e$.*

(c) *The relation $\leqslant$ on $\mathcal{BP}(kG)$ is a partial order which is respected by the conjugation action of G.*

Clearly $(\{1\}, b) \in \mathcal{BP}(kG, b)$ and Part (b) of the above theorem implies that if $(P, e) \in \mathcal{BP}(kG, b)$ then $(\{1\}, b) \leqslant (P, e)$. Parts (a) and (c) further imply that if $(Q, f) \leqslant (P, e)$ holds for elements in $\mathcal{BP}(kG)$ then $(Q, f) \in \mathcal{BP}(kG, b)$ if and only if $(P, e) \in \mathcal{BP}(kG, b)$.

For Brauer pairs $(Q, f), (P, e) \in \mathcal{BP}(kG)$ one writes $(Q, f) \trianglelefteq (P, e)$ if $Q \trianglelefteq P$, f is P-stable and $\mathrm{Br}_P(f)e = e$, cf. [AKO11, Definition IV.2.13]. The following result is well-known to specialists. We state it for convenient future reference and give a proof for the convenience of the reader.

2.2 Theorem *For $(Q, f), (P, e) \in \mathcal{BP}(kG)$ with $Q \leqslant P$ the following statements are equivalent:*

(i) *One has $(Q, f) \leqslant (P, e)$.*

(ii) *There exist primitive idempotents i of $(kG)^P$ and j of $(kG)^Q$ such that $ij = j = ji$, $\mathrm{Br}_P(i)e \neq 0$ and $\mathrm{Br}_Q(j)f \neq 0$.*

(iii) *There exist Brauer pairs $(Q_i, d_i) \in \mathcal{BP}(kG)$, $i = 0, \ldots, n$, such that*

$$(Q, f) = (Q_0, d_0) \trianglelefteq (Q_1, d_1) \trianglelefteq \cdots \trianglelefteq (Q_n, d_n) = (P, e)\,.$$

(iv) *For every primitive idempotent i of $(kG)^P$ with $Br_P(i)e \neq 0$ one has $Br_Q(i)f \neq 0$.*

(v) *There exists a primitive idempotent i of $(kG)^P$ such that $\mathrm{Br}_P(i)e \neq 0$ and $\mathrm{Br}_Q(i)f = \mathrm{Br}_Q(i) \neq 0$.*

(vi) *There exists a primitive idempotent i of $(kG)^P$ such that $Br_P(i)e \neq 0$ and $\mathrm{Br}_Q(i)f \neq 0$.*

Proof The equivalences (i) $\iff$ (ii) $\iff$ (iii) follow from [AKO11, Proposition IV.2.14]. Moreover, the implications (i)$\Rightarrow$(iv) and (v)$\Rightarrow$(vi) are trivial and the implication (i)$\Rightarrow$(v) follows from the fact that the image of a primitive idempotent under a surjective k-algebra homomorphism is either 0 or a primitive idempotent.

Next we show that (iv) implies (i). Let i be a primitive idempotent of $(kG)^P$ such that $\mathrm{Br}_P(i)e = \mathrm{Br}_P(i) \neq 0$. By (iv), $\mathrm{Br}_Q(i)f \neq 0$. By Theorem 2.1(a) there exists a block idempotent f' of $kC_G(Q)$ such that $(Q, f') \leqslant (P, e)$. Thus, $\mathrm{Br}_Q(i)f' = Br_Q(i)$ which implies that $0 \neq \mathrm{Br}_Q(i)f = \mathrm{Br}_Q(i)f'f$ and further that $f = f'$ and thus $(Q, f) \leqslant (P, e)$.

Finally, we show that (vi) implies (i). Let i be as in (vi). By Theorem 2.1(a) there exists a block idempotent f' of $kC_G(Q)$ such that $(Q, f') \leqslant (P, e)$. This implies $\mathrm{Br}_Q(i)f' = \mathrm{Br}_Q(i) \neq 0$ and $0 \neq \mathrm{Br}_Q(i)f = \mathrm{Br}_Q(i)f'f$. Thus $f = f'$ and $(Q, f) \leqslant (P, e)$. □

Recall that if $I \leqslant H \leqslant G$ then we have a well-defined trace map

$$\mathrm{Tr}_I^H \colon (kG)^I \to (kG)^H\,, \quad a \mapsto \sum_{x \in [H/I]} {}^x a\,.$$

A subgroup P of G, minimal with the property that $b \in \mathrm{Tr}_P^G((kG)^P)$, is called a *defect group* of the block idempotent b and of the block algebra kGb. The defect groups of kGb form a single

G-conjugacy class of p-subgroups of G. Maximal elements in $\mathcal{BP}(kG, b)$ enjoy properties that resemble the Sylow Theorem for finite groups.

2.3 Theorem *([AKO11, Theorem 2.20])* (a) *The maximal elements in $\mathcal{BP}(kG, b)$ with respect to $\leqslant$ form a single G-orbit.*

(b) *For $(P, e) \in \mathcal{BP}(kG, b)$ the following are equivalent.*

(i) *(P, e) is a maximal element in $\mathcal{BP}(kG, b)$.*

(ii) *P is a defect group of kGb.*

(iii) *P is maximal among all p-subgroups of G with the property $\mathrm{Br}_P(b) \neq 0$.*

3 Fusion systems of block algebras

Throughout this section, p is a prime. We first recall the basic notions and properties of fusion systems, a structure introduced by Puig. Our terminology follows [AKO11, Chapter I].

For subgroups Q and R of a finite group G we denote by $\mathrm{Hom}_G(Q, R)$ the set of all group homomorphisms $\varphi\colon Q \to R$ with the property that there exists $x \in G$ with $\varphi(u) = c_x(u)$ for all $u \in Q$. Moreover, we set $\mathrm{Aut}_G(Q) := \mathrm{Hom}_G(Q, Q)$.

3.1 Definition ([AKO11, Definition I.2.1]) Let P be a finite p-group. A subcategory $\mathcal{F}$ of the category of finite groups whose objects are the subgroups of P is called a *fusion system* over P if for any two subgroups Q and R of P, the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ has the following properties:

(i) $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R)$ and every element of $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ is injective.

(ii) For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, the group isomorphism $Q \to \varphi(Q)$, $u \mapsto \varphi(u)$, and its inverse are morphisms in $\mathcal{F}$.

For instance, if G is a finite group and P is a p-subgroup of G, we obtain a fusion system $\mathcal{F}_P(G)$ over P by setting $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) := \mathrm{Hom}_G(Q, R)$, for all subgroups Q and R of P. Note that the intersection of two fusion systems over P is again a fusion system and that a fusion system over P is determined by the isomorphisms it contains. Thus the smallest fusion system over a finite p-group P is the fusion system $\mathcal{F}_P(P)$.

3.2 Definition ([AKO11, Definition I.2.4]) Let $\mathcal{F}$ be a fusion system over a finite p-group P. A subgroup Q of P is called *fully $\mathcal{F}$-centralized* if $|C_P(Q)| \geqslant |C_P(Q')|$ for any subgroup Q' of P which is $\mathcal{F}$-isomorphic to Q. Similarly, Q is called *fully $\mathcal{F}$-normalized* if $|N_P(Q)| \geqslant |N_P(Q')|$ for any subgroup Q' of P which is $\mathcal{F}$-isomorphic to Q.

3.3 Definition ([AKO11, Definition I.2.2]) Let $\mathcal{F}$ be a fusion system on a finite p-group P and let $\varphi\colon Q \to R$ be an isomorphism in $\mathcal{F}$. One denotes by N_φ the set of all elements $y \in N_P(Q)$ for which there exists $z \in N_P(R)$ with the property $\varphi \circ c_y = c_z \circ \varphi\colon Q \to R$. Note that $QC_P(Q) \leqslant N_\varphi \leqslant N_P(Q)$ and that N_φ does not depend on $\mathcal{F}$, but only on φ and P.

If $\mathcal{F}$ is a fusion system over a finite p-group P and $Q \leqslant P$ then we set $\mathrm{Aut}_{\mathcal{F}}(Q) := \mathrm{Hom}_{\mathcal{F}}(Q, Q)$, a subgroup of the automorphism group of Q. The following definition of saturation goes back to Stancu and is an equivalent reformulation of the original definition, see [AKO11, Proposition I.9.3].

3.4 Definition A fusion system $\mathcal{F}$ over a p-group P is called *saturated* if the following two conditions hold.

(i) *Sylow axiom:* The group $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(ii) *Extension axiom:* For every $Q \leqslant P$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized there exists a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, P)$ whose restriction to Q equals φ.

For instance, if P is a Sylow p-subgroup of a finite group G then the fusion system $\mathcal{F}_P(G)$ is saturated (see [AKO11, Theorem 2.3]).

3.5 Definition Let G be a finite group, let k be a field of characteristic p, let b be a block idempotent of kG, and let (P, e) be a maximal (kG, b)-Brauer pair. We define a category $\mathcal{F}_{(P,e)}(kGb)$ as follows. First, for every $Q \leqslant P$ denote by e_Q the unique block idempotent of $kC_G(Q)$ with $(Q, e_Q) \leqslant (P, e)$. The objects of $\mathcal{F}_{(P,e)}(kGb)$ are the subgroups of P and for subgroups Q and R of P let $\mathrm{Hom}_{\mathcal{F}_{(P,e)}(kGb)}(Q, R)$ denote the set of group homomorphisms $\varphi\colon Q \to R$ such that there exists $x \in G$ with $\varphi(u) = c_x(u)$ for all $u \in Q$ and ${}^x(Q, e_Q) \leqslant (R, e_R)$. Composition in $\mathcal{F}_{(P,e)}(kGb)$ is the usual composition of functions.

3.6 Remark Let kG, b, and (P, e) be as in Definition 3.5.

(a) It is clear from the definition that $\mathcal{F}_{(P,e)}(kGb)$ is a fusion system over P.

(b) If kGb is the principal block of kG, then by Brauer's third main theorem, $\mathcal{F}_{(P,e)}(kGb)$ is equal to $\mathcal{F}_P(G)$ and P is a Sylow p-subgroup of G. Thus, $\mathcal{F}_{(P,e)}(kGb)$ is saturated in this case.

(c) Example 3.8 below shows that in general the Sylow axiom does not hold for $\mathcal{F}_{(P,e)}(kGb)$. But we will show in Theorem 6.2 that the extension axiom holds for $\mathcal{F}_{(P,e)}(kGb)$.

The following theorem was first proved by Puig. It follows from Theorem IV.3.2 and Proposition IV.3.14 in [AKO11]. See also [L18, Theorem 8.5.2] and note that there the terminology is different: Fusion systems in [L18] are defined to be saturated fusion systems in our terminology.

3.7 Theorem *Let kG, b, and (P, e) be as in Definition 3.5 and suppose that the k-algebra $kC_G(P)e$ is split, i.e., for every simple $kC_G(P)e$-module V one has a k-algebra isomorphism $\mathrm{End}_{kC_G(P)e}(V) \cong k$. Then the fusion system $\mathcal{F}_{(P,e)}(kGb)$ is saturated.*

We are grateful to Radha Kessar who suggested the following example to us.

3.8 Example Let $p = 2$, $k = \mathbb{F}_2$, the field with 2 elements, and $G := D_{24} = (C_3 \times C_4) \rtimes C_2$, the dihedral group with 24 elements, with C_2 acting by inversion on $C_3 \times C_4$. Let g denote a generator of C_3. Then $b := g+g^2$ is a block idempotent of $\mathbb{F}_2G$ and $(P, e) := (C_4, b)$ is a maximal $(\mathbb{F}_2G, b)$-Brauer pair. We have $\mathrm{Aut}_P(P) = \{1\}$, since P is abelian and an easy computation shows that $\mathrm{Aut}_{\mathcal{F}_{(P,e)}(\mathbb{F}_2Gb)}(P) \cong C_2$. Thus, the Sylow axiom does not hold for $\mathcal{F}_{(P,e)}(\mathbb{F}_2Gb)$ and therefore the fusion system $\mathcal{F}_{(P,e)}(\mathbb{F}_2Gb)$ is not saturated.

4 Extension of scalars

Throughout this section L/K denotes a finite Galois extension of fields of characteristic $p > 0$ and Γ denotes its Galois group. Moreover, G denotes a finite group.

Γ acts via K-algebra automorphisms on the group algebra LG and also on $Z(LG)$ by applying $\gamma \in \Gamma$ to the the coefficients of an element in LG. Thus, Γ permutes the block idempotents of LG and fixes the block idempotents b of KG. Since $\mathrm{Br}_P\colon (LG)^P \to LC_G(P)$

commutes with the Γ-action, Theorem 2.3 implies that any Γ-conjugate of b has the same defect groups as b. We denote by Γ_b the stabilizer of b in Γ and set

$$\tilde{b} := \sum_{\gamma\in[\Gamma/\Gamma_b]} {}^{\gamma}b\,.$$

Clearly, $\tilde{b}$ is an idempotent in $(Z(LG))^{\Gamma} = Z(KG)$. More precisely one has the following:

4.1 Proposition (a) *Let b be a block idempotent of LG. Then $\tilde{b} := \sum_{\gamma\in[\Gamma/\Gamma_b]} {}^{\gamma}b$ is a block idempotent of KG.*

(b) *The map $b \mapsto \tilde{b}$ induces a bijection between the set of Γ-orbits of block idempotents of LG and the set of block idempotents of KG.*

(c) *If b is a block idempotent of LG and $\tilde{b}$ is the block idempotent of KG associated to it as in (a) then b and $\tilde{b}$ have the same defect groups.*

Proof (a) By definition, $\tilde{b}$ is the sum of the distinct Γ-conjugates of b, thus an idempotent of $Z(KG)$. To see that $\tilde{b}$ is primitive in $Z(KG)$, assume that $\tilde{b} = c_1 + c_2$ for non-zero orthogonal idempotents $c_1, c_2 \in Z(KG)$ and let I_1 and I_2 denote the set of primitive idempotents of $Z(LG)$ that occur in a primitive decomposition of c_1 and c_2 in $Z(LG)$, respectively. Then I_1 and I_2 are disjoint and Γ-stable. On the other hand $I_1 \cup I_2$ is the single Γ-orbit of b. This is a contradiction.

(b) This is immediate from (a).

(c) Let P be a defect group of $\tilde{b}$. By Theorem 2.3, one has $\mathrm{Br}_P(\tilde{b}) \neq 0$ in $KC_G(P) \subseteq LC_G(P)$. Thus $0 \neq \mathrm{Br}_P(\tilde{b}) = \sum_{\gamma\in[\Gamma/\Gamma_b]} \mathrm{Br}_P({}^{\gamma}b)$ implies that some Γ-conjugate of b, and therefore also b, has a defect group Q containing P. Thus, $0 \neq \mathrm{Br}_Q(b) = \mathrm{Br}_Q(b\tilde{b}) = \mathrm{Br}_Q(b)\mathrm{Br}_Q(\tilde{b})$, which implies that $\mathrm{Br}_Q(\tilde{b}) \neq 0$ and therefore $|Q| \leqslant |P|$. This implies $P = Q$. □

Note that Γ acts on $\mathcal{BP}(LG)$ via

$$ {}^{\gamma}(P, e) = (P, {}^{\gamma}e)\,, \tag{1}$$

for $\gamma \in \Gamma$ and $(P, e) \in \mathcal{BP}(LG)$. Note that this action commutes with the G-action on $\mathcal{BP}(LG)$ so that we obtain an action of $\Gamma \times G$ on $\mathcal{BP}(LG)$. Moreover, since Br_P commutes with the action of Γ and since the G-action on LG commutes with the Γ-action on LG, $\Gamma \times G$ acts via poset isomorphisms on $\mathcal{BP}(LG)$. Thus, if b is a block idempotent of LG and $\gamma \in \Gamma$, the G-posets $\mathcal{BP}(LGb)$ and $\mathcal{BP}(LG\,{}^{\gamma}b)$ are isomorphic via (1) and $\Gamma_b \times G$ acts via poset automorphisms on $\mathcal{BP}(LGb)$.

In the next proposition we write $\leqslant_K$ and $\leqslant_L$ for the poset structures of $\mathcal{BP}(KG)$ and $\mathcal{BP}(LG)$, respectively. They are related as follows.

4.2 Proposition *For $(Q, f), (P, e) \in \mathcal{BP}(LG)$ with $Q \leqslant P$, the following are equivalent:*

(i) *One has $(Q, \tilde{f}) \leqslant_K (P, \tilde{e})$ in $\mathcal{BP}(KG)$.*

(ii) *There exists $\gamma \in \Gamma$ such that $(Q, f) \leqslant_L {}^{\gamma}(P, e)$ in $\mathcal{BP}(LG)$.*

Proof Assume first that (i) holds and let i be a primitive idempotent of $(KG)^P$ such that $\mathrm{Br}_P(i)\tilde{e} = \mathrm{Br}_P(i) \neq 0$. Then, by definition also $\mathrm{Br}_Q(i)\tilde{f} = \mathrm{Br}_Q(i) \neq 0$. Let J be a primitive decomposition of i in $(LG)^P$. Since $\mathrm{Br}_P(i)\tilde{e} \neq 0$, there exists $j \in J$ such that $\mathrm{Br}_P(j)\tilde{e} \neq 0$. Thus, there exists $\gamma \in \Gamma$ such that $\mathrm{Br}_P(j)\,{}^{\gamma}e \neq 0$. Since $\mathrm{Br}_P(j)$ is primitive in $LC_G(P)$, we have $\mathrm{Br}_P(j)\,{}^{\gamma}e = \mathrm{Br}_P(j)$. Let f' be the block idempotent of $LC_G(Q)$ such that $(Q, f') \leqslant_L$

$(P, {}^\gamma e) = {}^\gamma(P,e)$. Then, by Theorem 2.2 also $\mathrm{Br}_Q(j)f' = \mathrm{Br}_Q(j) \neq 0$. Thus $\mathrm{Br}_Q(j)f'\tilde{f} = \mathrm{Br}_Q(j)\mathrm{Br}_Q(i)f'\tilde{f} = \mathrm{Br}_Q(j)f'\mathrm{Br}_Q(i)\tilde{f} = \mathrm{Br}_Q(j)\mathrm{Br}_Q(i) = \mathrm{Br}_Q(j) \neq 0$ which implies that $f'\tilde{f} \neq 0$. This implies $f' = {}^\delta f$ for some $\delta \in \Gamma$. Thus ${}^\delta(Q,f) \leqslant_L {}^\gamma(P,e)$ and (ii) holds after applying δ^{-1}.

Next assume that $\gamma \in \Gamma$ with $(Q,f) \leqslant_L {}^\gamma(P,e)$. By Theorem 2.1(a) there exists a block idempotent f_1 of $LC_G(Q)$ such that $(Q,\tilde{f}_1) \leqslant_K (P,\tilde{e})$. Since we already proved that (i) implies (ii), there exists $\delta \in \Gamma$ such that $(Q,f_1) \leqslant_L {}^\delta(P,e)$. Thus we have $(Q, {}^{\gamma^{-1}}f) \leqslant_L (P,e)$ and also $(Q, {}^{\delta^{-1}}f) \leqslant_L (P,e)$. The uniqueness part of Theorem 2.2(a) now implies that f and f_1 are Γ-conjugate. Thus $\tilde{f} = \tilde{f}_1$ and $(Q,\tilde{f}) \leqslant_K (P,\tilde{e})$. ∎

The following corollaries are now immediate from Proposition 4.2.

4.3 Corollary *The map*

$$\mathcal{BP}(LG) \to \mathcal{BP}(KG)\,, \quad (P,e) \mapsto (P,\tilde{e})\,,$$

is a surjective morphism of G-posets, which restricts to a surjective morphism of G-posets $\mathcal{BP}(LGb) \to \mathcal{BP}(KG\tilde{b})$ for every block idempotent b of LG.

4.4 Corollary *Let b be a block idempotent of LG and let $(P,e) \in \mathcal{BP}(LGb)$ be a maximal LGb-Brauer pair. Then $(P,\tilde{e}) \in \mathcal{BP}(KG\tilde{b})$ is a maximal $(KG\tilde{b})$-Brauer pair and one obtains an inclusion of fusion systems*

$$\mathcal{F}_{(P,e)}(LGb) \to \mathcal{F}_{(P,\tilde{e})}(KG\tilde{b})$$

which is the identity on objects and on morphisms.

5 The Main Theorem

We keep p, G, L/K, and Γ as introduced at the beginning of Section 4. Moreover we fix a block idempotent b of LG and denote by Γ_b the stabilizer of b in Γ. We fix a maximal LGb-Brauer pair $(P,e) \in \mathcal{BP}(LGb)$. For every $Q \leqslant P$, let e_Q denote the unique block idempotent of $LC_G(Q)$ such that $(Q,e_Q) \leqslant (P,e)$ in $\mathcal{BP}(LG)$. By Proposition 4.2, one has $(Q,\widetilde{e_Q}) \leqslant (P,\tilde{e})$ so that $\widetilde{e_Q} = \tilde{e}_Q$. This allows us to use the notation $\tilde{e}_Q$ for both purposes. Recall that $\Gamma \times G$ acts on $\mathcal{BP}(LG)$ and $\Gamma_b \times G$ acts on $\mathcal{BP}(LGb)$ via poset isomorphisms. Note that for any $(Q,f) \in \mathcal{BP}(LGb)$ one has $\Gamma_{(Q,f)} = \Gamma_f$. For the stabilizer in G of a KG-Brauer pair or LG-Brauer pair (Q,f) we will write $N_G(Q,f)$.

Let $p_1\colon G\times\Gamma \to G$ and $p_2\colon G\times\Gamma\to\Gamma$ denote the projection maps. For any subgroup X of $G\times\Gamma$, we set $k_1(X) := \{g \in G \mid (g,1) \in X\}$ and $k_2(X) := \{\gamma \in \Gamma \mid (1,\gamma) \in X\}$. As explained in [B10, p. 24], one has

$$k_1(X) \trianglelefteq p_1(X) \leqslant G \quad \text{and} \quad k_2(X) \trianglelefteq p_2(X) \leqslant \Gamma \quad \text{with} \quad p_1(X)/k_1(X) \cong p_2(X)/k_2(X) \qquad (2)$$

via $gk_1(X) \leftrightarrow \gamma k_2(X)$ if and only if $(g,\gamma) \in X$.

We denote by $K(b)$ and $K(e)$ the subfields of L obtained by adjoining the coefficients of the block idempotents $b \in LG$ and $e \in LC_G(P)$. Thus, $K(b)$ is the fixed field of Γ_b in L and $K(e)$ is the fixed field of Γ_e in L.

5.1 Proposition *Let b be a block idempotent of LG.*

(a) *For any $(R, e_R) \leqslant (Q, e_Q)$ in $\mathcal{BP}(LGb)$ one has $\Gamma_e = \Gamma_{(P,e)} \leqslant \Gamma_{(Q,e_Q)} \leqslant \Gamma_{(R,e_R)} \leqslant \Gamma_{(\{1\},b)} = \Gamma_b$. In particular, $K(b) \subseteq K(e)$.*

(b) *Let $X := \mathrm{stab}_{G\times\Gamma}(P, e)$ be the stabilizer of the maximal LGb-Brauer pair (P, e). One has*

$$k_1(X) = N_G(P, e)\,, \quad p_1(X) = N_G(P, \tilde{e})\,, \quad k_2(X) = \Gamma_e\,, \quad \text{and} \quad p_2(X) = \Gamma_b\,.$$

(c) *One has $N_G(P, e) \trianglelefteq N_G(P, \tilde{e})$ and $N_G(P, \tilde{e})/N_G(P, e) \cong \Gamma_b/\Gamma_e$. Moreover, $K(e)/K(b)$ is a Galois extension with cyclic Galois group isomorphic to $N_G(P, \tilde{e})/N_G(P, e)$.*

Proof (a) It suffices to show that $\Gamma_{(Q,e_Q)} \leqslant \Gamma_{(R,e_R)}$. Let $\gamma \in \Gamma_{(Q,e_Q)}$. Then ${}^{\gamma}(R, e_R) \leqslant_L {}^{\gamma}(Q, e_Q) = (Q, {}^{\gamma}e_Q) = (Q, e_Q)$. The uniqueness part of Theorem 2.1(a) implies that ${}^{\gamma}e_R = e_R$. Thus, $\gamma \in \Gamma_{(R,e_R)}$.

(b) The first equation is clear from the definition of $k_1(X)$. For the proof of the second equation, let $g \in p_1(X)$. Then there exists $\gamma \in \Gamma$ with $(P, e) = {}^{(g,\gamma)}(P, e) = ({}^{g}P, {}^{g\gamma}e)$. From ${}^{g\gamma}e = e$ it follows that ${}^{g}\tilde{e} = \tilde{e}$. Thus ${}^{g}(P, \tilde{e}) = (P, \tilde{e})$ and $g \in N_G(P, \tilde{e})$. Conversely, if $g \in N_G(P, \tilde{e})$ then ${}^{g}\tilde{e} = \tilde{e}$ which implies that there exists $\gamma \in \Gamma$ with ${}^{g}e = {}^{\gamma}e$. Thus, ${}^{(g,\gamma^{-1})}(P, e) = (P, e)$ and $g \in p_1(X)$. The third equation follows immediately from the definition of $k_2(X)$. For the proof of the fourth equation let $\gamma \in p_2(X)$. Then there exists $g \in G$ with ${}^{(g,\gamma)}(P, e) = (P, e)$. Since $(\{1\}, b) \leqslant (P, e)$, this implies ${}^{(g,\gamma)}(\{1\}, b) \leqslant {}^{(g,\gamma)}(P, e) = (P, e)$. The uniqueness part in Theorem 2.1(a) implies that ${}^{(g,\gamma)}(\{1\}, b) = (1, b)$ and that $\gamma \in \Gamma_b$. Conversely, assume that $\gamma \in \Gamma_b$. Then $(\{1\}, b) \leqslant (P, e)$ implies $(\{1\}, b) = {}^{(1,\gamma)}(\{1\}, b) \leqslant {}^{(1,\gamma)}(P, e) = (P, {}^{\gamma}e)$. This implies that both (P, e) and ${}^{\gamma}(P, e)$ are maximal LGb-Brauer pairs. By Theorem 2.3(a), there exists $g \in G$ such that ${}^{g}(P, {}^{\gamma}e) = (P, e)$. Thus $(g, \gamma) \in X$ and $\gamma \in p_2(X)$.

(c) The assertions of the first sentence follow from Part (b) and (2). For the second statement it suffices to show that Γ_b/Γ_e is cyclic. Note that the coefficients of $e \in LC_G(P)$ generate a finite field extension of the prime field $\mathbb{F}_p$ in L, which we denote by $\mathbb{F}_p(e)$. Since $\Gamma_e \trianglelefteq \Gamma_b$, we have a Galois extension $K(e)/K(b)$ with Galois group $\Delta \cong \Gamma_b/\Gamma_e$. Now, restriction from $K(e)$ to $\mathbb{F}_p(e)$ is an injective group homomorphism from Δ to the cyclic Galois group $\mathrm{Gal}(\mathbb{F}_p(e)/\mathbb{F}_p)$. In fact, if $\delta \in \Delta$ restricts to the identity on $\mathbb{F}_p(e)$, then it is the identity on $\mathbb{F}_p(e)$ and on K, thus on $K(e)$. This completes the proof of Part (c). □

Next we give a more precise picture of the inclusion of fusion systems from Corollary 4.4. In the following theorem the term $\langle \mathcal{F}, \sigma \rangle$ denotes the fusion system *generated* by $\mathcal{F}$ and σ, i.e., the intersection of all fusion systems over P that contain $\mathcal{F}$ and σ.

5.2 Theorem *Let L/K be a finite Galois extension of fields of characteristic $p > 0$ with Galois group Γ, let b be a block idempotent of LG, and let (P, e) be a maximal LGb-Brauer pair. Set $\mathcal{F} := \mathcal{F}_{(P,e)}(LGb)$ and $\tilde{\mathcal{F}} := \mathcal{F}_{(P,\tilde{e})}(KG\tilde{b})$. Let $g_0 \in N_G(P, e)$ be such that $g_0 N_G(P, \tilde{e})$ generates $N_G(P, e)/N_G(P, \tilde{e})$ (see Proposition 5.1(c)) and set $\sigma := c_{g_0} \in \mathrm{Aut}(P)$. Then $\tilde{\mathcal{F}} = \langle \mathcal{F}, \sigma \rangle$.*

More precisely, $\sigma \in \mathrm{Aut}_{\tilde{\mathcal{F}}}(P)$ and, for any subgroups Q and R of P and any $\varphi \in \mathrm{Hom}_{\tilde{\mathcal{F}}}(Q, R)$, there exist $i \in \mathbb{Z}$, $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, \sigma^{-i}(R))$ and $\psi' \in \mathrm{Hom}_{\mathcal{F}}(\sigma^{i}(Q), R)$ with $\varphi = \sigma^{i}|_{\sigma^{-i}(R)} \circ \psi = \psi' \circ \sigma^{i}|_{Q}$.

Proof Since $g_0 \in N_G(P, \tilde{e})$, we have $\sigma = c_{g_0} \in \mathrm{Aut}_{\tilde{\mathcal{F}}}(P)$. It follows that $\langle \mathcal{F}, \sigma \rangle \subseteq \tilde{\mathcal{F}}$. In order to prove the reverse inclusion, let Q and R be subgroups of P and let $\varphi \in \mathrm{Hom}_{\tilde{\mathcal{F}}}(Q, R)$. Then there exists $g \in G$ such that $\varphi = c_g \colon Q \to R$ and ${}^{g}(Q, \tilde{e}_Q) \leqslant_K (R, \tilde{e}_R)$. By Proposition 4.2

there exists $\gamma \in \Gamma$ such that ${}^{g}(Q, e_Q) \leqslant_L (R, {}^{\gamma}e_R)$. Since $(\{1\}, b) = {}^{g}(\{1\}, b) \leqslant_L {}^{g}(Q, e_Q) \leqslant_L (R, {}^{\gamma}e_R)$ and also $(\{1\}, {}^{\gamma}b) \leqslant_L (R, {}^{\gamma}e_R)$, Theorem 2.1(a) implies $(\{1\}, b) = (\{1\}, {}^{\gamma}b)$ so that $\gamma \in \Gamma_b$. Thus, both (P, e) and $(P, {}^{\gamma}e)$ are maximal LGb-Brauer pairs. Theorem 2.3(a) implies that there exists $h \in G$ such that ${}^{h}(P, e) = (P, {}^{\gamma}e)$ and we obtain $(P, e) = {}^{h^{-1}}(P, {}^{\gamma}e) \geqslant_L {}^{h^{-1}}(R, {}^{\gamma}e_R) = ({}^{h^{-1}}R, {}^{h^{-1}\gamma}e_R)$. Again, Theorem 2.1(a) implies that ${}^{h^{-1}\gamma}e_R = e_{h^{-1}Rh}$ and therefore ${}^{h^{-1}g}(Q, e_Q) \leqslant_L {}^{h^{-1}}(R, {}^{\gamma}e_R) = ({}^{h^{-1}}R, e_{h^{-1}Rh})$. This in turn implies that the homomorphism $\alpha := c_{h^{-1}g} \colon Q \to {}^{h^{-1}}R$ belongs to $\mathrm{Hom}_{\mathcal{F}}(Q, {}^{h^{-1}}R)$ and that the homomorphism $\varphi = c_g \colon Q \to R$ factors as

$$\varphi = c_h \circ \alpha \colon Q \to {}^{h^{-1}}R \to R\,. \tag{3}$$

Since ${}^{h}(P, e) = (P, {}^{\gamma}e)$, we obtain $h \in N_G(P, \tilde{e})$ and can write $h = g_0^i x$ for some $i \in \mathbb{Z}$ and $x \in N_G(P, e)$. This implies that the map $c_h \colon P \to P$ factors as $c_h = \sigma^i \circ \beta \colon P \to P$ where $\sigma^i = c_{g_0^i} \colon P \to P$ and $\beta := c_x \in \mathrm{Aut}_{\mathcal{F}}(P)$, since $x \in N_G(P, e)$. Restriction to ${}^{h^{-1}}R$ yields the factorization

$$c_h|_{h^{-1}Rh} = \sigma^i|_{\beta(h^{-1}Rh)} \circ \beta|_{h^{-1}Rh} \colon {}^{h^{-1}}R \to \beta({}^{h^{-1}}R) \to R$$

with $\beta({}^{h^{-1}}R) = \sigma^{-i}(R)$ and $\beta|_{h^{-1}Rh} \in \mathrm{Hom}_{\mathcal{F}}({}^{h^{-1}}R, R)$. Setting $\psi := \beta|_{h^{-1}Rh} \circ \alpha \colon Q \to \sigma^{-1}(R)$ and using (3) we obtain the desired factorization of φ. This also implies the inclusion $\tilde{\mathcal{F}} \subseteq \langle \mathcal{F}, \sigma \rangle$.

In order to find ψ' with the desired property we use the elements g, h, x, and i from the first part of the proof and note that

$$(P, e) = {}^{\gamma^{-1}h}(P, e) \geqslant_L {}^{\gamma^{-1}}({}^{h}Q, {}^{h}e_Q) = ({}^{h}Q, {}^{\gamma^{-1}h}e_Q)\,,$$

which implies that ${}^{\gamma^{-1}h}e_Q = e_{hQh^{-1}}$. Thus,

$${}^{gh^{-1}}({}^{h}Q, e_{hQh^{-1}}) = {}^{gh^{-1}}({}^{h}Q, {}^{\gamma^{-1}h}e_Q) = ({}^{g}Q, {}^{g\gamma^{-1}}e_Q) \leqslant_L (R, e_R)\,,$$

which implies that $\alpha' := c_{gh^{-1}} \colon {}^{h}Q \to R$ belongs to $\mathrm{Hom}_{\mathcal{F}}({}^{h}Q, R)$. Thus, φ can be factored as

$$\varphi = c_g = c_{gh^{-1}} \circ c_h = \alpha' \circ c_h \colon Q \to {}^{h}Q \to R\,. \tag{4}$$

We can rewrite $h = g_0^i x = x' g_0^i$ for some $x' \in N_G(P, e)$ and obtain an element $\beta' \in \mathrm{Aut}_{\mathcal{F}}(P)$ together with a factorization $c_h = \beta' \circ \sigma^i \colon P \to P$. Restricting this equation to Q yields a factorization

$$c_h = \beta'|_{\sigma^i(Q)} \circ \sigma^i|_Q \colon Q \to \sigma^i(Q) \to {}^{h}Q\,.$$

Setting $\psi' := \alpha' \circ \beta'|_{\sigma^i(Q)} \in \mathrm{Hom}_{\mathcal{F}}(\sigma^i(Q), R)$, the factorization in (4) can now be expressed as $\varphi = \psi' \circ \sigma^i|_Q$ as claimed. □

6 Consequences of the Main Theorem

In this section we prove several consequences of Theorem 5.2.

Recall that if $\mathcal{F}$ is a fusion system over a p-group P, a subgroup Q of P is called $\mathcal{F}$-centric if $C_P(R) = Z(R)$ for all subgroups R of P which are $\mathcal{F}$-isomorphic to Q.

6.1 Proposition *Let L/K, b, (P,e) and $\mathcal{F} \subseteq \tilde{\mathcal{F}}$ be as in Theorem 5.2.*

(a) *A subgroup Q of P is fully $\mathcal{F}$-centralized if and only if it is fully $\tilde{\mathcal{F}}$-centralized.*

(b) *A subgroup Q of P is fully $\mathcal{F}$-normalized if and only if it is fully $\tilde{\mathcal{F}}$-normalized.*

(c) *A subgroup Q of P is $\mathcal{F}$-centric if and only it is $\tilde{\mathcal{F}}$-centric.*

Proof The 'if'-parts follow immediately from the fact that the $\mathcal{F}$-isomorphism class of Q is a subset of the $\tilde{\mathcal{F}}$-isomorphism class of Q. For the forward implications note that by Theorem 5.2 two subgroups Q and Q' of P are $\tilde{\mathcal{F}}$-isomorphic if and only if there exists a subgroup Q'' of P such that Q is $\mathcal{F}$-isomorphic to Q'' and $Q' = \sigma^i(Q'')$ for some $i \in \mathbb{Z}$. Moreover, $\sigma^i(C_P(Q'')) = C_P(\sigma^i(Q''))$, $\sigma^i(N_P(Q'')) = N_P(\sigma^i(Q''))$, and $\sigma^i(Z(Q'')) = Z(\sigma^i(Q''))$, since σ^i is an automorphism of P. The result is now immediate. □

The following Theorem is known to experts. See for instance the part of the proof of [L18, Theorem 8.5.2] dealing with the extension axiom and note that it does not use any assumptions on the field of coefficients k. Below is a proof with a different approach, using Theorem 5.2.

6.2 Theorem *Let k be a field of characteristic $p > 0$ and let c be a block idempotent of kG. Then the extension axiom holds for the fusion system of kGc, for any choice of maximal Brauer pair.*

Proof Let (P,f) be a maximal kGc-Brauer pair. We apply Theorem 5.2 with $K = k$, a splitting field L of $KC_G(P)f$ such that L/K is a finite Galois extension with Galois group Γ, and to a block idempotent b of LG with $cb \neq 0$. Then $c = \tilde{b}$. Moreover, there exists a maximal LGb-Brauer pair (P,e) such that $ef = e$ and therefore $f = \tilde{e}$. We aim to show that the fusion system $\tilde{\mathcal{F}} = \mathcal{F}_{(P,\tilde{e})}(KG\tilde{b})$ satisfies the extension axiom. Note that by Theorem 3.7, the extension axoim holds for $\mathcal{F} = \mathcal{F}_{(P,e)}(LGb)$, since L is a splitting field of $LC_G(P)e$. Let $\varphi \in \mathrm{Hom}_{\tilde{\mathcal{F}}}(Q,P)$ be such that $\varphi(Q)$ is fully $\tilde{\mathcal{F}}$-normalized. By Theorem 5.2 we can factorize $\varphi = \sigma^i \circ \psi$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$. With $\varphi(Q)$ also $\psi(Q) = \sigma^{-i}(\varphi(Q))$ is fully $\tilde{\mathcal{F}}$-normalized, since they are $\tilde{\mathcal{F}}$-isomorphic and $N_P(\psi(Q)) = \sigma^{-i}(N_P(\varphi(Q)))$. By Proposition 6.1(b), $\psi(Q)$ is fully $\mathcal{F}$-normalized. Since $\mathcal{F}$ satisfies the extension axiom, there exists $\hat{\psi} \in \mathrm{Hom}_{\mathcal{F}}(N_\psi, P)$ such that $\hat{\psi}|_Q = \psi$. It follows that $\hat{\varphi} := \sigma^i \circ \hat{\psi} \in \mathrm{Hom}_{\tilde{\mathcal{F}}}(N_\psi, P)$ extends φ. To finish the proof it suffices to show that $N_\varphi \subseteq N_\psi$. So let $x \in N_\varphi$. Then $x \in N_P(Q)$ and there exists $y \in N_P(\varphi(Q))$ with $\varphi \circ c_x = c_y \circ \varphi \colon Q \xrightarrow{\sim} \varphi(Q)$. But this implies

$$\psi \circ c_x = \sigma^{-i} \circ \varphi \circ c_x = \sigma^{-i} \circ c_y \circ \varphi = c_{\sigma^{-i}(y)} \circ \sigma^{-i} \circ \varphi = c_{\sigma^{-i}(y)} \circ \psi \,,$$

with $\sigma^{-i}(y) \in \sigma^{-i}(N_P(\varphi(Q))) = N_P(\sigma^{-i}(\varphi(Q))) = N_P(\psi(Q))$. Thus, $N_\varphi \subseteq N_\psi$ and the proof is complete. □

6.3 Theorem *Let L/K, b, (P,e) and $\mathcal{F} \subseteq \tilde{\mathcal{F}}$ be as in Theorem 5.2. The fusion system $\tilde{\mathcal{F}}$ is saturated if and only if the fusion system $\mathcal{F}$ is saturated and p does not divide $[N_G(P,\tilde{e}) : N_G(P,e)] = [\Gamma_b : \Gamma_e] = [K(e) : K(b)]$. In particular, if moreover L is a splitting field for $LC_G(P)e$, then $\tilde{\mathcal{F}}$ is saturated if and only if p does not divide $[N_G(P,\tilde{e}) : N_G(P,e)] = [\Gamma_b : \Gamma_e] = [K(e) : K(b)]$.*

Proof Note that the map $N_G(P,e) \to \mathrm{Aut}_{\mathcal{F}}(P)$, $g \mapsto c_g$ induces an isomorphism $N_G(P,e)/C_G(P) \to \mathrm{Aut}_{\mathcal{F}}(P)$ which maps $PC_G(P)/C_G(P)$ to $\mathrm{Aut}_P(P)$. Thus, the Sylow axiom holds for $\mathcal{F}$ if and only if $p \nmid [N_G(P,e) : PC_G(P)]$. Similarly, the Sylow axiom holds for $\tilde{\mathcal{F}}$ if and only if $p \nmid [N_G(P,\tilde{e}) : PC_G(P)]$. By Theorem 6.2 it suffices to show that the Sylow axiom holds for $\tilde{\mathcal{F}}$ if and only it holds for $\mathcal{F}$ and $p \nmid [\Gamma_b : \Gamma_e]$. But, by Proposition 5.1(c), one has $[\Gamma_b : \Gamma_e] = [N_G(P,\tilde{e}) : N_G(P,e)] = [K(e) : K(b)]$ which implies the result. □

Next we will show that a weak form of Alperin's fusion theorem holds for arbitrary block fusion systems.

6.4 Definition Let $\mathcal{F}$ be a fusion system over a p-group P. We say that *Alperin's weak fusion theorem holds for* $\mathcal{F}$ if $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(Q) \mid Q \in \mathcal{C}\rangle$, where $\mathcal{C}$ is the set of subgroups of P which are $\mathcal{F}$-centric and fully $\mathcal{F}$-normalized.

6.5 Theorem *Let k be a field of characteristic p and let c be a block idempotent of kG. Then Alperin's weak fusion theorem holds for the fusion system of kGc, for any choice of maximal kGc-Brauer pair.*

Proof Set $K := k$ and choose L, b, (P,e) as in the proof of Theorem 6.2 with $c = \tilde{b}$ and apply Theorem 5.2 to this situation with $\mathcal{F} := \mathcal{F}_{(P,e)}(LGb)$ and $\tilde{\mathcal{F}} := \mathcal{F}_{(P,\tilde{e})}(KG\tilde{b})$. We need to show that Alperin's weak fusion theorem holds for $\tilde{\mathcal{F}}$. Since $\mathcal{F}$ is saturated, Alperin's weak fusion theorem holds for $\mathcal{F}$, see for instance [L18, Theorem 8.2.8]. Thus, $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(Q) \mid Q \in \mathcal{C}\rangle$, where $\mathcal{C}$ denotes the set of subgroups of P which are $\mathcal{F}$-centric and fully $\mathcal{F}$-normalized. Moreover, by Proposition 6.1, $\mathcal{C}$ is equal to the set $\tilde{\mathcal{C}}$ of subgroups of P which are $\tilde{\mathcal{F}}$-centric and fully $\tilde{\mathcal{F}}$-normalized. Thus, by Theorem 5.2, we have

$$\tilde{\mathcal{F}} = \langle \mathcal{F}, \sigma\rangle = \langle \{\mathrm{Aut}_{\mathcal{F}}(Q) \mid Q \in \mathcal{C}\} \cup \{\sigma\}\rangle \subseteq \langle \mathrm{Aut}_{\tilde{\mathcal{F}}}(Q) \mid Q \in \mathcal{C}\rangle \subseteq \tilde{\mathcal{F}}\,.$$

But this implies $\tilde{\mathcal{F}} = \langle \mathrm{Aut}_{\tilde{\mathcal{F}}}(Q) \mid Q \in \mathcal{C}\rangle = \langle \mathrm{Aut}_{\tilde{\mathcal{F}}}(Q) \mid Q \in \tilde{\mathcal{C}}\rangle$, which means that Alperin's weak fusion theorem holds for $\tilde{\mathcal{F}}$. □

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