A note on blocks of finite groups with TI Sylow p-subgroups

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Abstract

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. We prove that functorial equivalence over $\mathbb{F}$ and perfect isometry between blocks of finite groups do not imply each other.

Keywords: block, perfect isometry, functorial equivalence.

MSC2020: 20C20, 20C34, 20J15.

Let $\mathbb{F}$ denote an algebraically closed field of characteristic 0 and let k denote an algebraically closed field of characteristic $p > 0$. In modular representation theory, there are different notions of equivalences between blocks of finite groups such as Puig equivalence, splendid Rickard equivalence, p-permutation equivalence, isotypies and perfect isometries ([B90], [BX08], [BP20]). Each equivalence in that list implies the subsequent one and they are related to prominent outstanding conjectures in modular representation theory, such as Broué's abelian defect group conjecture (Conjecture 9.7.6 in [L18]), Puig's finiteness conjecture (Conjecture 6.4.2 in [L18]) and Donovan's conjecture (Conjecture 6.1.9 in [L18]).

In [BY22] together with Bouc, we introduced another equivalence of blocks, namely functorial equivalences over R where R is a commutative ring. To each pair (G, b) of a finite group G and a block idempotent b of kG, we associate a canonical diagonal p-permutation functor over R. If (H, c) is another such pair, we say that (G, b) and (H, c) are *functorially equivalent over R* if their associated functors are isomorphic, see [BY22, Section 10].

We proved that the number of isomorphism classes of simple modules, the number of ordinary characters, and the defect groups are preserved under functorial equivalences over $\mathbb{F}$ ([BY22, Theorem 10.5]). Moreover we proved that for a given finite p-group D, there are only finitely many pairs (G, b), where G is a finite group and b is a block idempotent of kG with defect groups isomorphic to D, up to functorial equivalence over $\mathbb{F}$ ([BY22, Theorem 10.6]) and we provided a sufficient condition for two blocks to be functorially equivalent over $\mathbb{F}$ in the situation of Broué's abelian defect group conjecture ([BY22, Theorem 11.1]).

With Bouc, we also showed that if two blocks are p-permutation equivalent, then they are functorially equivalent over R, for any R ([BY22, Lemma 10.2(ii)]). The natural question hence is to understand the relation between functorial equivalences and isotypies and perfect isometries.

Remark 1 *For $p = 2$, the identity map induces a perfect isometry between the principal blocks of D_8 and Q_8, but the pairs $(D_8, 1)$ and $(Q_8, 1)$ are not functorially equivalent over $\mathbb{F}$ since D_8 is not isomorphic to Q_8.*

In this paper, we also give a negative answer to the question whether a functorial equivalence over $\mathbb{F}$ implies a perfect isometry by showing that the principal 2-block of a simple Suzuki group is functorially equivalent to its Brauer correspondent.

Theorem 2 *Let G be a finite group with TI Sylow p-subgroup P. Let b be a block idempotent of kG with a defect group P and let c be the block idempotent of $kN_G(P)$ which is in Brauer correspondence with b. Then the pairs (G, b) and $(N_G(P), c)$ are functorially equivalent over $\mathbb{F}$.*

Proof Since P is a TI Sylow p-subgroup, by a well-known result (see [L18, Theorem 9.8.6] for instance), the $(kGb, kN_G(P)c)$-bimodule $bkGc$ and its dual induce a stable equivalence of Morita type between kGb and $kN_G(P)c$. In particular, we have a stable p-permutation equivalence and hence a stable functorial equivalence over $\mathbb{F}$ between (G, b) and $(N_G(P), c)$. Moreover, by [BM90, Theorem 9.2], kGb and $kN_G(P)c$ have the same number of isomorphism classes of simple modules. Therefore, by [BY23, Theorem 1.2(i)], there is a functorial equivalence over $\mathbb{F}$ between (G, b) and $(N_G(P), c)$. □

Corollary 3 *For $p = 2$, the principal 2-block of the Suzuki group $\mathrm{Sz}(2^{2n+1})$, $n \geqslant 1$, is functorially equivalent over $\mathbb{F}$ to its Brauer correspondent. In particular, a functorial equivalence over $\mathbb{F}$ between blocks does not necessarily imply a perfect isometry.*

Proof The first assertion follows from Theorem 2 and the second assertion follows from the fact that the principal 2-block of $\mathrm{Sz}(2^{2n+1})$ is not perfectly isometric to its Brauer correspondent, see [C00] or [R00]. □

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