Diagonal p-permutation functors in characteristic p

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Abstract

Let p be a prime number. We consider diagonal p-permutation functors over a (commutative, unital) ring R in which all prime numbers different from p are invertible. We first determine the finite groups G for which the associated essential algebra $\mathcal{E}_{\mathsf{R}}(G)$ is non zero: These are groups of the form $G = L\langle u\rangle$, where (L, u) is a D^{Δ}-pair.

When R is an algebraically closed field $\mathbb{F}$ of characteristic 0 or p, this yields a parametrization of the simple diagonal p-permutation functors over $\mathbb{F}$ by triples (L, u, W), where (L, u) is a D^{Δ}-pair, and W is a simple $\mathbb{F}\mathrm{Out}(L, u)$-module.

Finally, we describe the evaluations of the simple functor $\mathsf{S}_{L,u,W}$ parametrized by the triple (L, u, W). We show in particular that if G is a finite group and $\mathbb{F}$ has characteristic p, the dimension of $\mathsf{S}_{L,1,\mathbb{F}}(G)$ is equal to the number of conjugacy classes of p-regular elements of G with defect isomorphic to L.

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1 Introduction

Let k be an algebraically closed field of positive characteristic p, and R be a commutative ring (with 1). As in [6] and [7], we consider the following category category $\mathsf{R}pp_k^{\Delta}$:

- The objects of $\mathsf{R}pp_k^{\Delta}$ are the finite groups.
- For finite groups G and H, the hom set $\mathrm{Hom}_{\mathsf{R}pp_k^{\Delta}}(G, H)$ is defined as $\mathsf{R}T^{\Delta}(H, G) = \mathsf{R} \otimes_{\mathbb{Z}} T^{\Delta}(H, G)$, where $T^{\Delta}(H, G)$ is the Grothendieck group of diagonal p-permutation (kH, kG)-bimodules.
- The composition in $\mathsf{R}pp_k^{\Delta}$ is induced by the usual tensor product of bimodules.
- The identity morphism of the group G is the class of the (kG, kG)-bimodule kG.

The category $\mathcal{F}_{\mathsf{R}pp_k}^{\Delta}$ of *diagonal p-permutation functors over* R is the category of R-linear functors from $\mathsf{R}pp_k^{\Delta}$ to the category R-Mod of all R-modules. It is an abelian category.

In [6] and [7], we mainly considered the case where R is an algebraically closed field $\mathbb{F}$ of characteristic 0. In [7], we showed in particular that the category $\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}$ is semisimple, and we classified and described its simple objects. For an arbitrary commutative ring R, we also introduced a new equivalence for blocks of groups algebras, called *functorial equivalence* over R, using diagonal p-permutation functors over R naturally attached to pairs (G, b) of a finite group G and a block idempotent b of kG. This led us in particular to prove the following finiteness theorem in the spirit of Donovan's and Puig's finiteness conjectures: For a given finite p-group D, there is only a finite number of pairs (G, b) of a finite group G and a block idempotent b of kG, with defect isomorphic to D, up to functorial equivalence over $\mathbb{F}$. We also showed that important invariants of blocks (like the number of simple modules or the number of ordinary irreducible characters) are preserved by functorial equivalence over $\mathbb{F}$, and we gave a characterization of nilpotent blocks in terms of functorial equivalence.

A natural question is then to see what happens when R is a field of *positive characteristic*, in particular when $\mathsf{R} = k$. An additional motivation for considering this case is that there are diagonal p-permutation functors attached to blocks, which vanish when considered over $\mathbb{F}$, but are of a fundamental interest when defined over k: For example, the Hochschild cohomology functors, sending a pair (G, b) as above to the i-th Hochschild cohomology group $HH^i(kGb, kGb)$ (or its k-dual, isomorphic to the i-th Hochschild homology group $HH_i(kGb, kGb)$). Let us mention here a question raised by Linckelmann ([11]): If b has non-trivial defect, is it true that $HH^1(kGb, kGb) \neq 0$?

This paper is a first step in the study of diagonal p-permutation functors in characteristic p, and we focus on simple functors. The main result we obtain (Theorem 5.25) is a parametrization and a description of these objects. In particular, we show how to compute the evaluations of such simple functors.

A key ingredient to the parametrization and description of simple functors is *the essential algebra* $\mathcal{E}_{\mathsf{R}}(G)$ of a group G, namely the quotient of the endomorphism algebra $\mathsf{R}T^{\Delta}(G, G)$ of G in the category $\mathsf{R}pp_k^{\Delta}$ by the ideal of linear combinations of endomorphisms which factor through a group of order strictly smaller than $|G|$. We first find some conditions on G (Corollary 3.4, Theorem 3.6, Theorem 3.7) for the (non-)vanishing of $\mathcal{E}_{\mathsf{R}}(G)$. In particular, if any prime number different from p is invertible in R, we show (Corollary 3.8) that $\mathcal{E}_{\mathsf{R}}(G)$ is non zero if and only if G is a semidirect product $L \rtimes \langle u \rangle$ (which we denote by $L\langle u \rangle$), where (L, u) is a D^{Δ}*-pair*, that is a pair of a finite p-group L and a p'-automorphism u of L (Definition 3.9). Moreover, we describe completely (Theorem 4.9) the structure of the algebra $\mathcal{E}_{\mathsf{R}}(G)$ in this case.

In Section 5, we study simple diagonal p-permutation functors, so we assume that R is a field $\mathbb{F}$ of characteristic 0 or p. Applying the results of the previous sections, we know that if S is a simple diagonal p-permutation functor over $\mathbb{F}$, then a minimal group for S is of the form $L\langle u \rangle$, where (L, u) is a D^{Δ}-pair, and the evaluation $V = S\big(L\langle u \rangle\big)$ is a simple $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$-module. Conversely, to any triple (L, u, V), where (L, u) is a D^Δ-pair and V is a simple $\mathcal{E}_\mathbb{F}(L\langle u\rangle)$-module, we associate a simple functor $S_{L\langle u\rangle,V}$ with minimal group $L\langle u\rangle$, and such that $S_{L\langle u\rangle,V}\big(L\langle u\rangle\big) \cong V$. Then we compute (Theorem 5.23) the evaluation $S_{L\langle u\rangle,V}(G)$ at an arbitrary finite group G.

The precise structure of the essential algebra given by Theorem 4.9 now allows for another parametrization of the simple functors, namely by triples (L, u, W), where (L, u) is a D^Δ-pair and W is a simple module for the algebra $\mathbb{F}\mathrm{Out}(L, u)$-module of the group $\mathrm{Out}(L, u)$ (Notation 3.10) of outer automorphisms of (L, u). In Theorem 5.25, we describe the evaluations of the simple functor $\mathsf{S}_{L,u,W}$ parametrized by such a triple (L, u, W).

Section 6 is devoted to some examples: First the simple functor $S_{\mathbf{1},1,\mathbb{F}}$, which turns out to be closely related to the Cartan map (Lemma 6.2, Proposition 6.8). This example shows in particular that the category $\mathcal{F}_{\mathbb{F}pp_k}$ is *not* semisimple when $\mathbb{F}$ has characteristic p. Then we describe (Theorem 6.11) the evaluations of the simple functor $\mathsf{S}_{L,1,W}$. In particular (Corollary 6.14), we show that for a finite group G, the dimension of $\mathsf{S}_{L,1,k}(G)$ is equal to the number of conjugacy classes of p'-elements of G with *defect isomorphic to* L (Definition 6.13).

2 Notation and terminology[1]

Throughout the paper:

- k is an algebraically closed field of positive characteristic p.
- R is a commutative ring (with 1).
- For a finite group G, we denote by $\mathsf{Proj}(kG)$ the group of projective kG-modules, and by $R_k(G)$ the Grothendieck group of the category of finite dimensional kG-modules. We set $\mathsf{RProj}(G) = \mathsf{R} \otimes_\mathbb{Z} \mathsf{Proj}(G)$ and $\mathsf{R}R_k(G) = \mathsf{R} \otimes_\mathbb{Z} R_k(G)$.
- If P is a p-subgroup of a finite group G, and M is a kG-module, we denote by $M[P]$ the Brauer quotient of M at P, and by $\mathrm{Br}_P : M^P \to M[P]$ the projection map. The module $M[P]$ is a $k\overline{N}_G(P)$-module, where $\overline{N}_G(P) = N_G(P)/P$.
- For a finite group G, a *p-permutation kG-module* (see [9]) is a direct summand of a permutation kG-module, i.e. of a module admitting a G-invariant k-basis. Equivalently, a kG-module M is a p-permutation module if the restriction $\mathrm{Res}^G_S M$ of M to a Sylow p-subgroup S of G is a permutation kS-module.
- From [9], we know that the indecomposable p-permutation kG-modules (up to isomorphism) are parametrized by pairs (P, E), where P is a p-subgroup

[1] An additional list of symbols is included at the end of the paper.

of G, up to conjugation, and E is an indecomposable projective $k\overline{N}_G(P)$-module, up to isomorphism. The indecomposable module $M(P,E)$ parametrized by the pair (P,E) is the only indecomposable direct summand with vertex P of $L_{P,E} = \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{\overline{N}_G(P)}^{N_G(P)} E$, the other direct indecomposable summands having vertex strictly contained in P, up to conjugation. The module $M(P,E)$ has vertex P, and $M(P,E)[P] \cong E$ as $k\overline{N}_G(P)$-modules.

- It follows that the Grothendieck group of p-permutation kG-modules, for relations given by direct sum decomposition, has a basis consisting of the modules $L_{P,E}$, where P is a p-subgroup of G, up to conjugation, and E is an indecomposable projective $k\overline{N}_G(P)$-module.

- When G and H are finite groups, and L is a subgroup of $H \times G$, we denote by $p_1(L)$ (resp. $p_2(L)$) the projection of L in H (resp. in G), and we set

$$k_1(L) = \{h \in H \mid (h,1) \in L\} \quad \text{and} \quad k_2(L) = \{g \in G \mid (1,g) \in L\}.$$

We say that L is *diagonal* if $k_1(L) = k_2(L) = 1$. Equivalently,

$$L = \Delta(Y,\pi,X) = \big\{\big(\pi(x),x\big) \mid x \in X\big\},$$

where X is a subgroup of G and $\pi : X \to Y$ is a group isomorphism from X to a subgroup Y of H. If $X = Y$ and $\pi = \mathrm{Id}$, we simply write $\Delta(X) = \Delta(X,\mathrm{Id},X)$. For $X \leq G$ and an embedding $\psi : X \hookrightarrow H$, we also write $\Delta_\psi(X)$ instead of $\Delta\big(\psi(X),\psi,X\big)$.

- For finite groups G and H, a p-permutation (kH,kG)-bimodule is a (kH,kG)-bimodule which is a p-permutation module when viewed as a $k(H \times G)$-module. A p-permutation (kH,kG)-bimodule M is *diagonal* if in addition M is projective when viewed as a left kH-module and a right kG-module. Equivalently M is a p-permutation (kH,kG)-bimodule, and all the vertices of the indecomposable summands of M are diagonal p-subgroups of $H \times G$.

- For finite groups G and H, we denote by $T^\Delta(H,G)$ the Grothendieck group of diagonal p-permutation (kH,kG)-bimodules, for relations given by direct sum decomposition. We set $\mathsf{R}T^\Delta(H,G) = \mathsf{R} \otimes_{\mathbb{Z}} T^\Delta(H,G)$. The group $T^\Delta(H,G)$ has a basis consisting of the bimodules of the form

$$\mathrm{Ind}_{N_{H\times G}(P)}^{H\times G} \mathrm{Inf}_{\overline{N}_{H\times G}(P)}^{N_{H\times G}(P)} E,$$

where P is a diagonal p-subgroup of $H \times G$ (up to conjugation), and E is an indecomposable projective $\overline{N}_{H\times G}(P)$-module.

- When G, H, and K are finite groups, if M is a diagonal p-permutation (kG,kH)-bimodule and N is a diagonal p-permutation (kK,kH)-bimodule, then $N \otimes_{kH} M$ is a diagonal p-permutation (kK,kG)-bimodule. This induces a well defined bilinear map

$$T^\Delta(K,H) \times T^\Delta(H,G) \to T^\Delta(K,G),$$

still denoted $(v,u)\mapsto v\otimes_{kH} u$. This bilinear map is also the composition in the category $\mathsf{R}pp_k^\Delta$ of the introduction, so it will be sometimes denoted by $(v,u)\mapsto v\circ u$.

- For finite groups G and H, we say that an element $u\in \mathsf{R}T^\Delta(G,H)$ is *right essential* (resp. *left essential*) if it cannot be factored through groups of order strictly smaller than $|H|$ (resp. of order strictly smaller than $|G|$), that is if $u\notin \sum_{|K|<|H|} \mathsf{R}T^\Delta(G,K)\circ \mathsf{R}T^\Delta(K,H)$ (resp. if $u\notin \sum_{|K|<|G|} \mathsf{R}T^\Delta(G,K)\circ \mathsf{R}T^\Delta(K,H)$). A (kG,kH)-bimodule M is called right essential over R - or simply right essential - (resp. left essential) if the element M of $\mathsf{R}T^\Delta(G,H)$ is right essential (resp. left essential).

  If $|G|=|H|$, being left essential is equivalent to being right essential, so we simply say *essential*.

- In particular, for a finite group G, the endomorphism algebra of G in the category $\mathsf{R}pp_k^\Delta$ is $\mathsf{R}T^\Delta(G,G)$. The *essential algebra* (over R) of G is the quotient

$$\mathcal{E}_{\mathsf{R}}(G)=\mathsf{R}T^\Delta(G,G)/\sum_{|H|<|G|}\mathsf{R}T^\Delta(G,H)\circ \mathsf{R}T^\Delta(H,G)$$

  of $\mathsf{R}T^\Delta(G,G)$ by the (two sided) ideal of non-essential elements. We denote by $u\mapsto \varepsilon_{\mathsf{R}}(u)$ the projection map $\mathsf{R}T^\Delta(G,G)\to\mathcal{E}_{\mathsf{R}}(G)$.

- The main reason for considering the previous essential algebra is the following: By standard results (see e.g. [5], Lemma 2.5 and Proposition 2.7), if S is a simple diagonal p-permutation functor over R, and if G is a group such that $V:=S(G)\neq 0$, then V is a simple $\mathsf{R}T^\Delta(G,G)$-module, and S is isomorphic to the unique simple quotient $S_{G,V}$ of the functor $L_{G,V}: H\mapsto \mathsf{R}T^\Delta(H,G)\otimes_{\mathsf{R}T^\Delta(G,G)}V$. Moreover, if G is a group of minimal order such that $S(G)\neq 0$, then in fact V is a simple $\mathcal{E}_{\mathsf{R}}(G)$-module, and $S\cong S_{G,V}$. So we are looking for pairs (G,V) of a finite group G and a simple $\mathcal{E}_{\mathsf{R}}(G)$-module. In particular, for such a pair, the essential algebra $\mathcal{E}_{\mathsf{R}}(G)$ is non-zero.

- An elementary group (or Brauer elementary group) is a finite group of the form $Q\times C$, where Q is a q-group for some prime number q, and C is a cyclic group (that can be assumed of order prime to q). When p is a prime number, an elementary p'-group is an elementary group of order prime to p.

3 Vanishing of $\mathcal{E}_{\mathsf{R}}(G)$

Let G be a finite group. We want to know when the essential algebra $\mathcal{E}_{\mathsf{R}}(G)$ is non-zero. We start with some classical lemmas.

Lemma 3.1: *Let $G=P\rtimes K$, where P and K are finite groups of coprime order. Let moreover φ be an automorphism of G. Then:*

1. $\varphi(P) = P$.

2. $C_G(P) = Z(P)C_K(P)$.

3. Suppose that $C_G(P) = Z(P)$*, or equivalently by Assertion 2, that* K *acts faithfully on* P*. Then the following are equivalent:*

   (a) The restriction of φ *to* P *is the identity.*

   (b) φ *is an inner automorphism* i_w *of* G*, for some* $w \in Z(P)$*.*

Proof: 1. This is clear, since P is the set of elements of G of order dividing the order of P.

2. The inclusion $Z(P)C_K(P) \leq C_G(P)$ is clear. Conversely, if $xt \in C_G(P)$, where $x \in P$ and $t \in K$, then ${}^t y = y^x$ for any $y \in P$. It follows that ${}^{t^n} y = y^{x^n}$ for any $n \in \mathbb{N}$ and any $y \in P$. Taking $n = |x|$ gives $t^n \in C_K(P)$, hence $t \in C_K(P)$ since $(n, |t|) = 1$. Then $x \in C_P(P) = Z(P)$.

3. It is clear that (b) implies (a)[2]. For the converse, assume that (a) holds, and that $C_G(P) = Z(P)$. Let $x, y \in P$ and $s, t \in K$. Then

$$\begin{aligned}\varphi(xs \cdot yt) &= \varphi(x\,{}^s y \cdot st) = x\,{}^s y\varphi(st)\\ &= \varphi(xs)\varphi(yt) = x\varphi(s)y\varphi(t) = x\,{}^{\varphi(s)}y\,\varphi(s)\varphi(t).\end{aligned}$$

Hence ${}^s y = {}^{\varphi(s)} y$ for any $y \in P$. In other words $z(s) := s^{-1}\varphi(s) \in C_G(P)$, so z is a map from K to $Z(P)$.

Now $\varphi(s) = sz(s)$, so $z(st) = z(s)^t z(t)$ for any $s, t \in K$. In other words, the map z is a crossed morphism from K to $Z(P)$. Since K and $Z(P)$ have coprime order, it follows that there exists $w \in Z(P)$ such that $z(s) = w^s \cdot w^{-1}$, for any $s \in K$. In other words $\varphi(s) = s \cdot w^s \cdot w^{-1} = wsw^{-1} = i_w(s)$. Since $i_w(x) = x = \varphi(x)$ for any $x \in P$, it follows that $\varphi = i_w$. □

Lemma 3.2: *Let* G *be a finite group, and* P *be a normal* p*-subgroup of* G*. Then:*

1. $N_{G\times G}\big(\Delta(P)\big) = \{(a, b) \in G \times G \mid ab^{-1} \in C_G(P)\}$.

2. Set $N = N_{G\times G}\big(\Delta(P)\big)$ *and* $\overline{N} = N/\Delta(P)$*. There is an isomorphism of* (kG, kG)*-bimodules*

$$kG \cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^N kC_G(P),$$

where the action of $\overline{N}$ *on* $kC_G(P)$ *is given by* $(a, b)\Delta(P) \cdot \gamma = a\gamma b^{-1}$*.*

[2] and we don't need the assumption $C_G(P) = Z(P)$ for that…

Proof: 1. This is clear, since $(a, b) \in N$ if and only if $x^a = x^b$, i.e. $x^{ab^{-1}} = x$, for all $x \in P$.

2. The group $\overline{N}$ permutes the set $C_G(P)$ transitively, and the stabilizer in $\overline{N}$ of $1 \in C_G(P)$ is the group $\{(a, a)\Delta(P) \mid a \in G\} = \Delta(G)/\Delta(P)$. So $kC_G(P) \cong \mathrm{Ind}_{\Delta(G)/\Delta(P)}^{\overline{N}} k$, and

$$\begin{aligned}\mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N} kC_G(P) &\cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N}\mathrm{Ind}_{\Delta(G)/\Delta(P)}^{\overline{N}} k\\ &\cong \mathrm{Ind}_N^{G\times G}\mathrm{Ind}_{\Delta(G)}^{N}\mathrm{Inf}_{\Delta(G)/\Delta(P)}^{\Delta(G)} k\\ &\cong \mathrm{Ind}_{\Delta(G)}^{G\times G} k \cong kG\end{aligned}$$

as (kG, kG)-bimodules. □

The next step is an important reduction allowed by the following stronger version of Dress induction theorem, due to Boltje and Külshammer ([2], Theorem 3.3):

Theorem 3.3: *Let H be a finite group, and U be an indecomposable kH-module with vertex D and source Z. Then, in the Green ring of kH, we have*

$$[U] = \sum_{i=1}^{n} a_i[\mathrm{Ind}_{H_i}^{G} V_i],$$

where, for $i = 1, \ldots, n$:

- *a_i is an integer.*
- *H_i is a subgroup of H such that $D_i := O_p(H_i) \leq D$ and H_i/D_i is an elementary p'-group.*
- *V_i is an indecomposable kH_i-module with vertex D_i and source $\mathrm{Res}_{D_i}^{H_i} V_i$, which is a direct summand of $\mathrm{Res}_{D_i}^{D} Z$.*

Corollary 3.4: *Let G be a finite group. Then $\mathcal{E}_{\mathsf{R}}(G) = 0$ unless $G \cong P \rtimes K$, where P is a p-group, and K is an elementary p'-group.*

Proof: We apply Theorem 3.3 to the case $H = G \times G$ and $U = kGb$, where b is a block idempotent of kG. Then U is a diagonal p-permutation (kG, kG)-bimodule with diagonal vertex $\Delta(D) = \Delta(D, \mathrm{Id}, D)$, where $D \leq G$ is a defect group of b. We can conclude that $[U]$ is a linear combination with integer coefficients of (isomorphism classes of) induced bimodules $\left[\mathrm{Ind}_{H_i}^{G\times G} V_i\right]$, where H_i is a subgroup of $G \times G$ such that $D_i = O_p(H_i) \leq \Delta(D)$ and $H_i/O_p(H_i)$ is an elementary p'-group. Let G_i be the first projection of H_i on G. Then the bimodule $\mathrm{Ind}_{H_i}^{G\times G}V_i$ factors as

$$\mathrm{Ind}_{H_i}^{G\times G}V_i \cong \mathrm{Ind}_{\Delta(G_i)}^{G\times G_i} \otimes_{kG_i} \mathrm{Ind}_{H_i}^{G_i\times G}V_i,$$

where H_i on the right hand side is viewed as a subgroup of $G_i \times G$. Now if $G_i < G$, then the image of kGb in $\mathcal{E}_{\mathsf{R}}(G)$ is equal to 0. And if $G_i = G$, then G is a quotient of H_i, so $G/O_p(G)$ is an elementary p'-group. In other words $G \cong P \rtimes K$, where P is a p-group, and K is an elementary p'-group.

Now $\mathcal{E}_{\mathsf{R}}(G)$ is non zero if and only if its identity element is non zero, that is if the image of the bimodule kG in $\mathcal{E}_{\mathsf{R}}(G)$ is non zero. Since kG is the direct sum of the bimodules kGb, when b runs through block idempotents of kG, there is at least one such idempotent b such that the image of kGb in $\mathcal{E}_{\mathsf{R}}(G)$ is non-zero. Hence $G \cong P \rtimes K$, where P is a p-group and K is an elementary p'-group. □

Lemma 3.5: *Let $G = P \rtimes K$, where P is a finite p-group, and K is an elementary p'-group. Let H be a finite group, and U be a right essential indecomposable diagonal p-permutation (kG, kH)-bimodule. Then:*

1. *The essential algebra $\mathcal{E}_{\mathsf{R}}(H)$ is non-zero. In particular $H = Q \rtimes L$, where Q is a p-group and L is an elementary p'-group.*

2. *There exist an injective group homomorphism $\pi : Q \hookrightarrow P$, a subgroup T of $N_{K\times L}\big(\Delta_\pi(Q)\big)$ with $p_2(T) = L$, and a simple kT-module W such that*

   $$U \cong \mathrm{Ind}_{\Delta_\pi(Q)\cdot T}^{G\times H}\mathrm{Inf}_T^{\Delta_\pi(Q)\cdot T}W$$

   as (kG, kH)-bimodules.

Proof: 1. If $\mathcal{E}_{\mathsf{R}}(H) = 0$, the identity (kH, kH)-bimodule kH factors through groups of order strictly smaller than $|H|$, so the same holds for U (by right composition with kH). Hence $\mathcal{E}_{\mathsf{R}}(H) \neq 0$, and Assertion 1 follows from Corollary 3.4.

2. From Assertion 1 follows in particular that the group $G \times H$ is solvable, with a normal Sylow p-subgroup $P \times Q$. Then there is a diagonal p-subgroup $\Delta_\pi(S)$ of $G \times H$, where S is a p-subgroup of H (that is, a subgroup of Q) and $\pi : S \hookrightarrow P$ is an injective group homomorphism, such that

$$U \cong \mathrm{Ind}_N^{G\times H}\mathrm{Inf}_{\overline{N}}^N E,$$

where $N = N_{G\times H}\big(\Delta_\pi(S)\big)$ and $\overline{N} = N/\Delta_\pi(S)$, and E is an indecomposable projective $k\overline{N}$-module.

Now $\overline{N}$ itself also has a normal Sylow p-subgroup X, and there is a p'-subgroup $\overline{T}$ of $\overline{N}$ such that $\overline{N} = X \rtimes \overline{T}$. Moreover $\overline{T}$ lifts to a p'-subgroup T of

N, that we can assume contained in the p'-Hall subgroup $K \times L$ of $G \times H$, up to replacing $\Delta_\pi(S)$ by a conjugate subgroup. Finally $E \cong \mathrm{Ind}_{\overline{T}}^{\overline{N}} \overline{W}$, where $\overline{W}$ is a simple $k\overline{T}$-module. Let W be the simple kT-module corresponding to $\overline{W}$ via the isomorphism $\overline{T} \cong T$. It follows that

$$\begin{aligned} U &\cong \mathrm{Ind}_N^{G\times H} \mathrm{Inf}_{\overline{N}}^N \mathrm{Ind}_{\overline{T}}^{\overline{N}} \overline{W} \\ &\cong \mathrm{Ind}_N^{G\times H} \mathrm{Ind}_{\Delta_\pi(S)\cdot T}^N \mathrm{Inf}_T^{\Delta_\pi(S)\cdot T} W \\ &\cong \mathrm{Ind}_{\Delta_\pi(S)\cdot T}^{G\times H} \mathrm{Inf}_T^{\Delta_\pi(S)\cdot T} W. \end{aligned}$$

Since U is right essential, we have $p_2\big(\Delta_\pi(S)\cdot T\big) = H = Q \cdot L$. This forces $S = Q$, and $p_2(T) = L$, proving Assertion 2. □

Theorem 3.6: *Let G be a finite group of the form $G = P \rtimes K$, where P is a p-group and K is a non-cyclic elementary p'-group. Then $|K|^2\, \mathcal{E}_{\mathsf{R}}(G) = 0$. In particular, if $|K|$ is invertible in R, then $\mathcal{E}_{\mathsf{R}}(G) = 0$.*

Proof: Let M be an essential indecomposable diagonal p-permutation (kG, kG)-bimodule. By Lemma 3.5, applied to $H = G$ and $U = M$, we know that

$$M \cong \mathrm{Ind}_{\Delta_\pi(P)\cdot T}^{G\times G} \mathrm{Inf}_T^{\Delta_\varphi(P)\cdot T} W,$$

for some $\pi \in \mathrm{Aut}(P)$, some subgroup T of $N_{K\times K}\big(\Delta_\pi(P)\big)$ with $p_2(T) = K$, and some simple kT-module W.

Now $T \leq K \times K$, so T is a p'-group, and $|T|$ divides $|K|^2$. Moreover by Artin's induction theorem, in $R_k(T) \cong R_{\mathbb{C}}(T)$, we have an equality of the form

$$|T|W = \sum_{i=1}^n n_i \mathrm{Ind}_{C_i}^T k_{\lambda_i},$$

where, for $1 \leq i \leq n$, C_i is a cyclic subgroup of T, n_i is an integer, and k_{λ_i} is a one dimensional kC_i-module. Hence in $\mathsf{R}T^\Delta(G, G)$, we have that

$$\begin{aligned} |T|M &= \sum_{i=1}^n n_i \mathrm{Ind}_{\Delta_\pi(P)\cdot T}^{G\times G} \mathrm{Inf}_T^{\Delta_\pi(P)\cdot T} \mathrm{Ind}_{C_i}^T k_{\lambda_i} \\ &= \sum_{i=1}^n n_i \mathrm{Ind}_{\Delta_\pi(P)\cdot T}^{G\times G} \mathrm{Ind}_{\Delta_\pi(P)\cdot C_i}^{\Delta_\pi(P)\cdot T} \mathrm{Inf}_{C_i}^{\Delta_\pi(P)\cdot C_i} k_{\lambda_i} \\ &= \sum_{i=1}^n n_i \mathrm{Ind}_{\Delta_\pi(P)\cdot C_i}^{G\times G} \mathrm{Inf}_{C_i}^{\Delta_\pi(P)\cdot C_i} k_{\lambda_i}. \end{aligned}$$

The image of $|K|^2 M$ in $\mathcal{E}_{\mathsf{R}}(G)$ is equal to the image of

$$\frac{|K|^2}{|T|}\sum_{i=1}^{n} n_i \mathrm{Ind}_{\Delta_\pi(P)\cdot C_i}^{G\times G}\mathrm{Inf}_{C_i}^{\Delta_\pi(P)\cdot C_i} k_{\lambda_i},$$

which is equal to zero unless there exists $i \in \{1,\ldots,n\}$ such that

$$p_1\big(\Delta_\varphi(P)\cdot C_i\big) = p_2\big(\Delta_\varphi(P)\cdot C_i\big) = G = P\cdot K.$$

This implies that K is a quotient of C_i. Hence K is cyclic, which completes the proof. □

Theorem 3.7: *Let $G = P \rtimes K$, where P is a p-group and K is a cyclic p'-group. If $C_K(P) \neq 1$, then $\mathcal{E}_{\mathsf{R}}(G) = 0$.*

Proof: [3] Since G has a normal Sylow p-subgroup, all the blocks of G have defect P. Moreover, if b is a block idempotent of G, then b is a linear combination of p-regular elements of $C_G(P) = Z(P)\times C_K(P)$, so $b \in kC_K(P)$.

Since $C_K(P) \leq Z(G)$, it means that the block idempotents of kG are exactly the primitive idempotents of the (split semisimple commutative) algebra $kC_K(P)$. Let e be one of them, and $k_\lambda = kC_K(P)e$ be the corresponding (one dimensional) simple $kC_K(P)$-module, where $\lambda : C_K(P) \to k^\times$ is the associated group homomorphism.

Let $\varpi : G \to K$ denote the projection map. Set $N = N_{G\times G}\big(\Delta(P)\big)$ and $\overline{N} = N/\Delta(P)$. There is a short exact sequence

$$1 \longrightarrow Z(P) \xrightarrow{\ i\ } \overline{N} \xrightarrow{\ s\ } \widetilde{K} \longrightarrow 1,$$

where

- $\widetilde{K} = \{(a,b)\in K\times K \mid a^{-1}b \in C_K(P)\}$.
- i is the map sending $z \in Z(P)$ to $(z,1)\Delta(P) \in \overline{N}$.
- s is the map sending $(a,b)\Delta(P)$ to $\big(\varpi(a),\varpi(b)\big)$.

So $\overline{N} \cong Z(P) \rtimes \widetilde{K}$, with the explicit embedding $\widetilde{K} \to \overline{N}$ sending $(a,b)\in\widetilde{K}$ to $(a,b)\Delta(P) \in \overline{N}$. We consider $\widetilde{K}$ as a subgroup of $\overline{N}$ via this embedding.

The Brauer quotient of the (kG,kG)-bimodule kG at $\Delta(P)$ is isomorphic to $kC_G(P)$, so $kGe[\Delta(P)] \cong kC_G(P)\mathrm{Br}_P(e) = kC_G(P)e = kZ(P)\otimes_k k_\lambda$, since $\mathrm{Br}_P(e) = e$ as $e \in kC_K(P)$. It follows that

$$kGe \cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N} kC_G(P)e \cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N}\big(kZ(P)\otimes_k k_\lambda\big),$$

[3] This proof is a slightly simplified and generalized version of the proof given by M. Ducellier in Proposition 4.1.2 of his thesis [10] in the case $\mathsf{R} = \mathbb{C}$.

where $\overline{N} \cong Z(P) \rtimes \widetilde{K}$ acts on $kZ(P) \otimes_k k_\lambda$ by

$$(a,b)\Delta(P).(z \otimes 1) = (ab^{-1})_p\, z \otimes (ab^{-1})_{p'} \cdot 1 = \lambda\big((ab^{-1})_{p'}\big)(ab^{-1})_p\, z \otimes 1$$

for $(a,b) \in N$ and $z \in Z(P)$, where $(ab^{-1})_p \in Z(P)$ and $(ab^{-1})_{p'} \in C_K(P)$ are the p-part and p'-part of $ab^{-1} \in C_G(P) = Z(P) \times C_K(P)$, respectively. Then $kZ(P) \otimes_k k_\lambda$ is isomorphic to $\mathrm{Ind}_{\widetilde{K}}^{\overline{N}} k_{\widetilde{\lambda}}$, where $\widetilde{\lambda} : \widetilde{K} \to k^\times$ sends $(a,b) \in \widetilde{K}$ to $\lambda(ab^{-1}) \in k^\times$. So

$$\begin{aligned} kGe &\cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N}\mathrm{Ind}_{\widetilde{K}}^{\overline{N}} k_{\widetilde{\lambda}} \\ &\cong \mathrm{Ind}_N^{G\times G}\mathrm{Ind}_{\Delta(P)\widetilde{K}}^{N}\mathrm{Inf}_{\widetilde{K}}^{\Delta(P)\widetilde{K}} k_{\widetilde{\lambda}} \\ &\cong \mathrm{Ind}_{\Delta(P)\widetilde{K}}^{G\times G}\mathrm{Inf}_{\widetilde{K}}^{\Delta(P)\widetilde{K}} k_{\widetilde{\lambda}}. \end{aligned} \tag{3.7.1}$$

Now we set $\overline{G} = G/C_K(P)$. We denote by $g \mapsto \overline{g}$ the projection map, and by $\delta : G \to G \times \overline{G}$ the map $g \mapsto (g, \overline{g})$. We will show that the (kG, kG)-bimodule kGe factors through the group $\overline{G}$, that is, there is a diagonal p-permutation $(kG, k\overline{G})$-bimodule U and a diagonal p-permutation $(k\overline{G}, kG)$-bimodule V such that $kGe \cong U \otimes_{k\overline{G}} V$.

The group P embeds in $G \times \overline{G}$ via δ. Its image $\delta(P)$ is a diagonal subgroup of $G \times \overline{G}$, and its normalizer is

$$N_\delta := N_{G\times\overline{G}}\big(\delta(P)\big) = \big\{(a, \overline{b}) \in G \times \overline{G} \mid \overline{x^a} = \overline{x^b},\ \forall x \in P\big\}.$$

In other words $(a, \overline{b}) \in N_\delta$ if and only if $\overline{x^{ab^{-1}}} = \overline{x}$ for all $x \in P$, or equivalently if the commutator $[x, ab^{-1}]$ is in $C_K(P)$. But since $P \trianglelefteq G$, we have that $[P, ab^{-1}] \subseteq P$. Hence $(a, \overline{b})$ normalizes $\delta(P)$ if and only if $[P, ab^{-1}] \subseteq P \cap C_K(P) = 1$, i.e. $ab^{-1} \in C_G(P)$. Thus

$$N_\delta = \{(a, \overline{b}) \in G \times \overline{G} \mid ab^{-1} \in Z(P) \times C_K(P)\}.$$

Recall that $\varpi : G \to K$ denotes the projection map. We have a surjective group homomorphism $\sigma : N_\delta \to K$ sending $(a, \overline{b})$ to $\varpi(a)$. It induces a surjective group homomorphism

$$\overline{\sigma} : \overline{N}_\delta := N_\delta/\delta(P) \to K$$

sending $(a, \overline{b})\delta(P)$ to $\varpi(a)$. The kernel of this morphism consists of the elements $(a, \overline{b})\delta(P)$ such that $a \in P$ and $ab^{-1} \in Z(P) \times C_K(P)$. Since

$$(a, \overline{b})\delta(P) = (1, \overline{ba^{-1}})(a, \overline{a})\delta(P) = (1, \overline{ba^{-1}})\delta(P),$$

and since $\overline{ba^{-1}} \in C_G(P)/C_K(P) = Z(P)$, we get a short exact sequence

$$1 \longrightarrow Z(P) \xrightarrow{\ \iota\ } \overline{N}_\delta \xrightarrow{\ \overline{\sigma}\ } K \longrightarrow 1$$

where $\iota(z) = (1, \overline{z})\delta(P)$ for $z \in Z(P)$. This sequence is split, via the morphism $a \in K \mapsto (a, \overline{a})\delta(P)$, for $a \in K$, so $\overline{N}_\delta \cong Z(P) \rtimes K$.

Now since K is cyclic, we can extend $\lambda : C_K(P) \to k^\times$ to a group homomorphism $\beta : K \to k^\times$. This gives a one dimensional kK-module k_β, that we can induce to $\overline{N}_\delta = Z(P) \rtimes K$. We get a projective $k\overline{N}_\delta$-module, and we set

$$U := \mathrm{Ind}_{N_\delta}^{G\times\overline{G}} \mathrm{Inf}_{\overline{N}_\delta}^{N_\delta} \mathrm{Ind}_K^{\overline{N}_\delta} k_\beta.$$

This is a diagonal p-permutation $(kG, k\overline{G})$-bimodule, and

$$U \cong \mathrm{Ind}_{N_\delta}^{G\times\overline{G}} \mathrm{Ind}_{\delta(P)K}^{N_\delta} \mathrm{Inf}_K^{\delta(P)K} k_\beta \cong \mathrm{Ind}_{\delta(P)K}^{G\times\overline{G}} \mathrm{Inf}_K^{\delta(P)K} k_\beta,$$

where K is viewed as a subgroup of N_δ via the map $a \in K \mapsto (a, \overline{a}) \in N_\delta$. We observe that $\delta(P)K$ is equal to $\delta(G)$, so

$$U \cong \mathrm{Ind}_{\delta(G)}^{G\times\overline{G}} \mathrm{Inf}_K^{\delta(G)} k_\beta.$$

We define similarly a $(k\overline{G}, kG)$-bimodule V by

$$V := \mathrm{Ind}_{\delta^o(G)}^{\overline{G}\times G} \mathrm{Inf}_{K^o}^{\delta^o(G)} k_{\beta^{-1}},$$

where $\delta^o : G \to \overline{G} \times G$ sends x to $(\overline{x}, x)$, and $K^o = \{(\overline{a}, a) \mid a \in K\}$.

Now we compute the tensor product $U \otimes_{k\overline{G}} V$ using Theorem 1.1 of [3]. Since $p_2\big(\delta(G)\big) = \overline{G}$, there is a single double coset $p_2\big(\delta(G)\big)\backslash\overline{G}/p_1\big(\delta^o(G)\big)$. Moreover $k_2\big(\delta(G)\big) = 1$, so we have

$$U \otimes_{k\overline{G}} V \cong \mathrm{Ind}_{\delta(G)*\delta^o(G)}^{G\times G} \big(\mathrm{Inf}_K^{\delta(G)} k_\beta \otimes_k \mathrm{Inf}_{K^o}^{\delta^o(G)} k_{\beta^{-1}}\big), \qquad (3.7.2)$$

where

$$\begin{aligned} \delta(G) * \delta^o(G) &= \big\{(a, b) \in G \times G \mid \exists \overline{c} \in \overline{G},\ (a, \overline{c}) \in \delta(G) \text{ and } (\overline{c}, b) \in \delta^o(G)\big\} \\ &= \big\{(a, b) \in G \times G \mid \overline{a} = \overline{b}\big\} \\ &= \big\{(a, b) \in G \times G \mid ab^{-1} \in C_K(P)\big\}. \end{aligned}$$

Now if $a = xu$ and $b = yv$, with $x, y \in P$ and $u, v \in K$, we have

$$ab^{-1} = xuv^{-1}y^{-1} = x \cdot {}^{uv^{-1}}(y^{-1}) \cdot uv^{-1},$$

so $ab^{-1} \in C_K(P)$ if and only if $uv^{-1} \in C_K(P)$ and $x = y$, i.e. $(u, v) \in \widetilde{K}$ and $(x, y) \in \Delta(P)$. It follows that $\delta(G) * \delta^o(G) = \Delta(P)\widetilde{K}$.

The action of $(a, b) \in \delta(G) * \delta^o(G)$ on the tensor product

$$T := \mathrm{Inf}_K^{\delta(G)} k_\beta \otimes_k \mathrm{Inf}_{K^o}^{\delta^o(G)} k_{\beta^{-1}}$$

is obtained as follows: Let $\overline{c} \in \overline{G}$ such that $(a, \overline{c}) \in \delta(G)$ and $(\overline{c}, b) \in \delta^o(G)$, that is $\overline{c} = \overline{a} = \overline{b}$. Then for $v \in \mathrm{Inf}_K^{\delta(G)} k_\beta$ and $w \in \mathrm{Inf}_{K^o}^{\delta^o(G)} k_{\beta^{-1}}$, we have

$(a,b)\cdot(v\otimes w)=(a,\overline{c})\cdot v\otimes(\overline{c},b)\cdot w$. Here T is one dimensional, with basis $1\otimes 1$, and

$$\begin{aligned}(a,b)\cdot(1\otimes 1)&=\beta\big(\varpi(a)\big)\beta\big(\varpi(b)\big)^{-1}(1\otimes 1)\\&=\beta\big(\varpi(ab^{-1})\big)(1\otimes 1)\\&=\lambda\big(\varpi(ab^{-1})\big)(1\otimes 1),\end{aligned}$$

since the restriction of β to $C_K(P)$ is equal to λ.

Now for $(a,b)\in\delta(G)*\delta^o(G)=\Delta(P)\widetilde{K}$, we have $\lambda\big(\varpi(ab^{-1})\big)=\widetilde{\lambda}\big(\rho(a,b)\big)$, where $\rho:\Delta(P)\widetilde{K}\to\widetilde{K}$ is the projection map. Then $T\cong\mathrm{Inf}_{\widetilde{K}}^{\Delta(P)\widetilde{K}}k_{\widetilde{\lambda}}$, and by 3.7.1 and 3.7.2, we get that

$$U\otimes_{k\overline{G}}V\cong\mathrm{Ind}_{\delta(G)*\delta^o(G)}^{G\times G}T\cong\mathrm{Ind}_{\Delta(P)\widetilde{K}}^{G\times G}\mathrm{Inf}_{\widetilde{K}}^{\Delta(P)\widetilde{K}}k_{\widetilde{\lambda}}\cong kGe,$$

as was to be shown.

So if $C_K(P)\neq 1$, all the bimodules kGe are mapped to 0 in $\mathcal{E}_{\mathsf{R}}(G)$, and the image of their direct sum kG is also 0. Now as its identity element is equal to 0, the algebra $\mathcal{E}_{\mathsf{R}}(G)$ itself is equal to 0. □

Corollary 3.8: *Let $G=P\rtimes K$, where K is an elementary p'-group of order invertible in* R*. If $\mathcal{E}_{\mathsf{R}}(G)\neq 0$, then K is cyclic and acts faithfully on P.*

Proof: Indeed if $\mathcal{E}_{\mathsf{R}}(G)\neq 0$, we know by Theorem 3.6 that K is cyclic, and by Theorem 3.7, that K acts faithfully on P. □

In other words $G=P\rtimes\langle u\rangle$, where (P,u) is a D^Δ-pair, as defined hereafter:[4]

Definition 3.9: *A D^Δ-pair is a pair (P,u) of a finite p-group P and a p'-automorphism u of P.*

We recall the following notation ([7], Notation 6.8):

Notation 3.10: *For a D^Δ-pair (L,u), we denote by $\mathrm{Aut}(L,u)$ the group of automorphisms of the semidirect product $L\langle u\rangle=L\rtimes\langle u\rangle$ which send u to a conjugate of u, and by $\mathrm{Out}(L,u)$ the quotient $\mathrm{Aut}(L,u)/\mathrm{Inn}(L\langle u\rangle)$ of this group by the group of inner automorphisms of $L\langle u\rangle$.*

[4] D^Δ-pairs were first introduced in the slightly different Definition 4.4 of [6]. The subsequent Lemma 4.5 there showed that the two definitions are equivalent.

4 Generators and relations for $\mathcal{E}_{\mathsf{R}}(G)$

Notation 4.1: *For a finite group G, denote by $G^\natural$ the group $\mathrm{Hom}(G,k^\times)$. For $\lambda \in G^\natural$, let k_λ denote the corresponding one dimensional kG-module. For $\gamma \in \mathrm{Aut}(G)$, denote*[5] *by $kG_{\gamma,\lambda}$ the (kG,kG)-bimodule equal to kG as a vector space, with action given by*

$$\forall (x,y,z) \in G^3,\ x\cdot z\cdot y = \lambda(y)^{-1}xz\gamma(y).$$

Lemma 4.2: *Let G be a finite group.*

1. *Let $\gamma \in \mathrm{Aut}(G)$ and $\lambda \in G^\natural$. Then $kG_{\gamma,\lambda}$ is a diagonal p-permutation bimodule, and there is an isomorphism of (kG,kG)-bimodules*
$$kG_{\gamma,\lambda} \cong \mathrm{Ind}_{\Delta_\gamma(G)}^{G\times G} k_\lambda.$$
2. [6] *Let $\delta \in \mathrm{Aut}(G)$ and $\mu \in G^\natural$. Then there is an isomorphism of (kG,kG)-bimodules*
$$kG_{\gamma,\lambda} \otimes_{kG} kG_{\delta,\mu} \cong kG_{\gamma\circ\delta,(\lambda\circ\delta)\times\mu}.$$
3. *If kG has only one block, then $kG_{\gamma,\lambda}$ is an indecomposable (kG,kG)-bimodule, for any $\gamma \in \mathrm{Aut}(G)$ and any $\lambda \in G^\natural$.*

Proof: 1. Let S be a Sylow p-subgroup of G. Then $\lambda(x)=1$ for any $x \in S$, so the restriction of $kG_{\gamma,\lambda}$ to the Sylow p-subgroup $S\times S$ of $G\times G$ is the permutation bimodule kG, with action $x\cdot z\cdot y = xz\gamma(y)$, for $x,y\in S$ and $z \in G$. Moreover this action is free on both sides, so $kG_{\gamma,\lambda}$ is a diagonal p-permutation bimodule. Finally, the map $g\in G \mapsto (g,1)\Delta_\gamma(G)$ is a bijection from G to $(G\times G)/\Delta_\gamma(G)$, and using this bijection, it is easy to check that $kG_{\gamma,\lambda} \cong \mathrm{Ind}_{\Delta_\gamma(G)}^{G\times G} k_\lambda$.

2. The map $(g\otimes g') \mapsto \lambda(g')^{-1}g\gamma(g')$, for $g,g' \in G$, from $kG_{\gamma,\lambda}\otimes_{kG} kG_{\delta,\mu}$ to $kG_{\gamma\circ\delta,(\lambda\circ\delta)\times\mu}$ induces a well defined isomorphism of (kG,kG)-bimodules.

3. It is clear that for any (kG,kG)-bimodule M, any $\delta \in \mathrm{Aut}(G)$, and any $\mu \in G^\natural$, the k-vector space $M\otimes_{kG} kG_{\delta,\mu}$ is isomorphic to M. So if $kG_{\gamma,\lambda}$ splits as a direct sum of non-zero (kG,kG)-bimodules M and M', the tensor product $kG_{\gamma,\lambda}\otimes_{kG} kG_{\delta,\mu}$, splits as the direct sum of (kG,kG)-bimodules $M\otimes kG_{\delta,\mu}$ and $M'\otimes kG_{\delta,\mu}$, none of which is equal to zero. Then $kG_{\gamma\circ\delta,(\lambda\circ\delta)\times\mu}$ splits as a direct sum of non-zero bimodules. Taking $\delta=\gamma^{-1}$ and $\mu = (\lambda\circ\gamma^{-1})^{-1}$, we get that the (kG,kG)-bimodule $kG_{\mathrm{Id},1} \cong kG$ splits non-trivially. So kG is not indecomposable, that is, kG has more than one block. □

In the rest of this section, in view of Corollary 3.8, we assume the following:

[5]This generalizes Definition 4.1.3 of [10], up to replacing λ with λ^{-1}, which is more convenient in our setting.

[6]Up to the previous change of notation, this is Proposition 4.1.4 in [10].

Hypothesis 4.3: *The group G is of the form $P \rtimes K$, where P is a p-group and K is a cyclic p'-group of order invertible in R, acting faithfully on P.*

We want to find the structure of the algebra $\mathcal{E}_{\mathsf{R}}(G)$. First we look for generators of $\mathcal{E}_{\mathsf{R}}(G)$ as an R-module.

Lemma 4.4: *Assume that 4.3 holds. Then:*

1. *kG has only one block. So for $\gamma \in \mathrm{Aut}(G)$ and $\lambda \in G^\natural$, the bimodule $kG_{\gamma,\lambda}$ is indecomposable, with vertex $\Delta_\gamma(P)$.*
2. *Let $\gamma, \delta \in \mathrm{Aut}(G)$ and $\lambda, \mu \in G^\natural$. Then the bimodules $kG_{\gamma,\lambda}$ and $kG_{\delta,\mu}$ are isomorphic if and only if $\lambda = \mu$ and $\delta \circ \gamma^{-1}$ is an inner automorphism of G.*
3. *Conversely, if M is an indecomposable diagonal p-permutation bimodule with vertex $\Delta_\gamma(P)$, for $\gamma \in \mathrm{Aut}(G)$, then there exists $\lambda \in G^\natural$ such that $M \cong kG_{\gamma,\lambda}$.*
4. *In particular, if M is an essential indecomposable diagonal p-permutation (kG, kG)-bimodule, there exist $\gamma \in \mathrm{Aut}(G)$ and $\lambda \in G^\natural$ such that $M \cong kG_{\gamma,\lambda}$ as (kG, kG)-bimodule.*

Proof: 1. The p-subgroup P of G is normal, and $C_G(P) = Z(P) \times C_K(P) = Z(P) \leq P$ since $C_K(P) = 1$ as K acts faithfully on P. So kG has only one block ([1] Proposition 6.2.2). Then all the bimodules $kG_{\gamma,\lambda}$ are indecomposable, by Lemma 4.2.

Let $\gamma \in \mathrm{Aut}(G)$. Set $N := N_{G\times G}\big(\Delta_\gamma(P)\big)$, and $\overline{N} = N/\Delta_\gamma(P)$. Then

$$N = \{(a, b) \in G \times G \mid a^{-1}\gamma(b) \in C_G(P) = Z(P)\},$$

so the second projection p_2 induces a short exact sequence

$$1 \longrightarrow Z(P) \longrightarrow \overline{N} \longrightarrow K \longrightarrow 1,$$

which is split (by the map $x \in K \mapsto \big(\gamma(x), x\big)\Delta_\gamma(P)$). Now the indecomposable projective $k\overline{N}$-modules are the modules $\mathrm{Ind}_K^{\overline{N}} k_\lambda$, for $\lambda \in K^\natural$. Moreover

$$\begin{aligned}\mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N}\mathrm{Ind}_K^{\overline{N}} k_\lambda &\cong \mathrm{Ind}_N^{G\times G}\mathrm{Ind}_{\Delta_\gamma(P)\cdot\Delta_\gamma(K)}^{N} k_\lambda\\ &\cong \mathrm{Ind}_{\Delta_\gamma(G)}^{G\times G} k_\lambda \cong kG_{\gamma,\lambda},\end{aligned}$$

by Lemma 4.2. This gives another proof that $kG_{\gamma,\lambda}$ is indecomposable, and also shows that it has vertex $\Delta_\gamma(P)$, and Brauer quotient

$$kG_{\gamma,\lambda}[\Delta_\gamma(P)] \cong \mathrm{Ind}_K^{\overline{N}} k_\lambda \cong kZ(P) \otimes_k k_\lambda. \tag{4.4.1}$$

2. First, if $\lambda = \mu$ and $\delta = i_x \circ \gamma$, where i_x is conjugation by $x \in G$, then one checks easily that the map $g \in G \mapsto gx \in G$ induces a bimodule isomorphism $kG_{\gamma,\lambda} \cong kG_{\delta,\mu}$.

For the converse, let $\gamma' = \gamma^{-1}$ and $\lambda' = (\lambda \circ \gamma')^{-1}$. Then by Lemma 4.2, we have an isomorphism of (kG, kG)-bimodules

$$kG_{\gamma,\lambda} \otimes_{kG} kG_{\gamma',\lambda'} \cong kG_{\gamma\circ\gamma',(\lambda\circ\gamma')\times\lambda'} = kG_{\mathrm{Id},1} = kG.$$

So if $kG_{\delta,\mu} \cong kG_{\gamma,\lambda}$, we have an isomorphism of (kG, kG)-bimodules

$$kG_{\delta,\mu} \otimes_{kG} kG_{\gamma',\lambda'} \cong kG,$$

that is

$$kG_{\delta\circ\gamma',(\mu\circ\gamma')\times\lambda'} \cong kG_{\mathrm{Id},1}.$$

So if we know that an isomorphism of bimodules $kG_{\theta,\rho} \cong kG_{\mathrm{Id},1}$, where $\theta \in \mathrm{Aut}(G)$ and $\rho \in G^\natural$ implies that θ is inner and $\rho = 1$, we are done, since we can conclude that $\delta \circ \gamma' = \delta \circ \gamma^{-1}$ is inner, and that $(\mu \circ \gamma') \times \lambda' = 1$, i.e. $\mu = \lambda$. In other words, we can assume $\gamma = \mathrm{Id}$ and $\lambda = 1$, and that $kG_{\delta,\mu} \cong kG$.

Now if $kG_{\delta,\mu}$ is isomorphic to kG, then its vertex $\Delta_\delta(P)$ is contained in - hence equal to - $\Delta(P)$, up to conjugation in $G\times G$. It means that the restriction of δ to P is equal to the conjugation by some element of G. Up to composing δ with some inner automorphism of G, we can assume that this restriction is equal to the identity, and then δ is equal to the conjugation by some element of $Z(P)$, by Lemma 3.1. This shows that δ is inner.

Then we have a bimodule isomorphism $kG_{\delta,\lambda} \cong kG_{\mathrm{Id},\lambda}$ by the first remark in the proof of Assertion 2. In other words, we can assume $\delta = \mathrm{Id}$ and $kG_{\mathrm{Id},\lambda} \cong kG_{\mathrm{Id},1}$. Then the Brauer quotients at $\Delta(P)$ of these bimodules are isomorphic. Hence $kZ(P) \otimes k_\lambda \cong kZ(P) \otimes k \cong kZ(P)$. Now the fixed points of $Z(P)$ on $kZ(P) \otimes k_\lambda$ form a kK-module isomorphic to k_λ, so $k_\lambda \cong k$ as K-module, hence $\lambda = 1$, as was to be shown.

3. Suppose conversely that M is an indecomposable diagonal p-permutation (kG, kG)-bimodule with vertex $\Delta_\gamma(P)$, where $\gamma \in \mathrm{Aut}(G)$. Then $M[\Delta_\gamma(P)]$ is an indecomposable projective $k\overline{N}$-module, of the form $\mathrm{Ind}_K^{\overline{N}} k_\lambda$ for some $\lambda \in K^\natural$, and then

$$M \cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N}\mathrm{Ind}_K^{\overline{N}} k_\lambda \cong kG_{\gamma,\lambda}.$$

4. As in the proof of Theorem 3.6, we apply Lemma 3.5, in the case $H = G$ and $U = M$. We know that

$$M \cong \mathrm{Ind}_{\Delta_\pi(P)\cdot T}^{G\times G}\mathrm{Inf}_T^{\Delta_\varphi(P)\cdot T} W,$$

for some $\pi \in \mathrm{Aut}(P)$, some subgroup T of $N_{K\times K}\big(\Delta_\pi(P)\big)$ with $p_2(T) = K$, and some simple kT-module W.

Moreover $k_1(T) \leq k_1\Big(N_{G\times G}\big(\Delta_\pi(P)\big)\Big) = C_G(P) = Z(P)$. So $k_1(T) = 1$ since T is a p'-group. Then $p_2 : T \to K$ is an isomorphism, with inverse θ, and $T = \Delta_\theta(K)$. Then T is cyclic, and $W \cong k_\lambda$ for some $\lambda \in T^\natural$.

Now T normalizes $\Delta_\pi(P)$ if and only if ${}^{\theta(x)}\pi(y) = \pi({}^x y)$ for any $x \in K$ and any $y \in P$. Then the map $\gamma : y \cdot x \mapsto \pi(y)\cdot\theta(x)$, where $y \in P$ and $x \in K$, is an automorphism of G, such that $\gamma(K) = K$, and $\Delta_\pi(P)\cdot L = \Delta_\gamma(G)$. Then

$$M \cong \mathrm{Ind}_{\Delta_\gamma(G)}^{G\times G} k_\mu,$$

where $\mu = \lambda \circ \varpi \in G^\natural \cong \Delta_\gamma(G)^\natural$. Now $M \cong kG_{\gamma,\mu}$ by Lemma 4.2. □

It follows that $\mathcal{E}_{\mathsf{R}}(G)$ is linearly generated by the images of the (kG, kG)-bimodules $kG_{\gamma,\lambda}$, for $\gamma \in \mathrm{Aut}(G)$ and $\lambda \in G^\natural$. By Lemma 4.4, we can take γ in a set of representatives of elements of $\mathrm{Out}(G)$. We want to describe the linear relations between these generators. In other words, we want to find equalities of the form

$$\sum_{\substack{\gamma\in\mathrm{Out}(G)\\ \lambda\in G^\natural}} r_{\gamma,\lambda}\, kG_{\gamma,\lambda} = \sum_{i=1}^n s_i\, U_i \otimes_{kH_i} V_i, \tag{4.4.2}$$

in $\mathsf{R}T^\Delta(G, G)$, where $r_{\gamma,\lambda} \in \mathsf{R}$, $n \in \mathbb{N}$, and for $1 \leq i \leq n$, H_i is a finite group of order smaller than the order of G, U_i is a diagonal p-permutation (kG, kH_i)-bimodule, V_i is a diagonal p-permutation (kH_i, kG)-bimodule, and $s_i \in \mathsf{R}$. We can assume moreover that for $1 \leq i \leq n$, the essential algebra $\mathcal{E}_{\mathsf{R}}(H_i)$ is non-zero: Indeed otherwise, the identity bimodule $kH_i \in \mathsf{R}T^\Delta(H_i, H_i)$ is a linear combination with coefficients in R of elements of $\mathsf{R}T^\Delta(H_i, X) \otimes_{kX} \mathsf{R}T^\Delta(X, H_i)$, for $|X| < |H_i|$, and we can replace H_i by smaller groups in (4.4.2).

Hence, by Corollary 3.4, we can assume that for $1 \leq i \leq n$, we have $H_i = Q_i \rtimes L_i$, where Q_i is a p-group, and L_i is an elementary p'-group. We can also assume that U_i and V_i are indecomposable, and that U_i is right essential and V_i is left essential.

Lemma 4.5: *Assume that 4.3 holds. Let H be a finite group, and U be a right essential indecomposable diagonal p-permutation (kG, kH)-bimodule. Then the vertices of U have order at most $|P|$. If U has vertex of order $|P|$, then there exists an injective group homomorphism $\sigma : H \hookrightarrow G$ such that $P \leq \sigma(H)$ and $\lambda \in H^\natural = \Delta_\sigma(H)^\natural$ such that*

$$U \cong \mathrm{Ind}_{\Delta_\sigma(H)}^{G\times H} k_\lambda.$$

Proof: We know from Lemma 3.5 that $H = Q \rtimes L$, where Q is a p-group with an embedding $\pi : Q \hookrightarrow P$, and L is an elementary p'-group. Moreover there is a subgroup T of $N_{K\times L}\big(\Delta_\pi(Q)\big)$ with $p_2(T) = L$, and a simple kT-module W such that

$$U \cong \mathrm{Ind}_{\Delta_\pi(Q)\cdot T}^{G\times H} \mathrm{Inf}_T^{\Delta_\pi(Q)\cdot T} W.$$

Then a vertex of U is contained in $\Delta_\pi(Q)\cdot T$ up to conjugation, so it has order at most $|Q|\le|P|$. And if it has order $|P|$, the embedding $\pi: Q\hookrightarrow P$ is an isomorphism. Moreover $k_1(T)\le k_1\Big(N_{G\times H}\big(\Delta_\pi(P)\big)\Big)=C_G(P)$, so $k_1(T)\le C_K(P)=1$ since K acts faithfully on P. Then the projection map $p_2: T\to L$ is an isomorphism, and then $T=\Delta_\tau(L)$, for some injective group homomorphism $\tau: L\hookrightarrow K$. In particular L and T are cyclic.

Moreover since $T=\Delta_\tau(L)\le N_{G\times H}\big(\Delta_\pi(P)\big)$, we have ${}^{\tau(l)}\pi(x)=\pi({}^l x)$ for any $l\in L$ and $x\in Q$. Then the map $\sigma: H=Q\cdot L\to G$ sending $x\cdot l$, for $x\in Q$ and $l\in L$, to $\pi(x)\cdot\tau(l)$, is an injective group homomorphism, such that $P\le\gamma(H)\le G$. Moreover $\Delta_\pi(Q)\cdot T=\Delta_\sigma(H)$. Finally T is cyclic, so there is $\lambda\in T^\natural=L^\natural\cong H^\natural$ such that $W=k_\lambda$, and $U\cong\operatorname{Ind}_{\Delta_\sigma(H)}^{G\times H}k_\lambda$, completing the proof. □

Theorem 4.6: *Assume that 4.3 holds. Let H be a finite group, let U (resp. V) be a right (resp. left) essential indecomposable diagonal p-permutation (kG,kH)-bimodule (resp. (kH,kG)-bimodule).*

1. *If $U\otimes_{kH}V$ has an indecomposable direct summand with vertex of order $|P|$, then there is a subgroup $I\cong H$ of G, containing P, an automorphism ψ of I, and $\zeta\in I^\natural=\Delta_\psi(I)^\natural$ such that*

$$U\otimes_{kH}V\cong\operatorname{Ind}_{\Delta_\psi(I)}^{G\times G}k_\zeta.$$

2. *If $U\otimes_{kH}V$ admits an essential indecomposable summand, then this summand has vertex $\Delta_\gamma(P)$ for some $\gamma\in\operatorname{Aut}(G)$, and there exists $J\le K$ and $\theta\in J^\natural$ such that $P\cdot J\cong H$ and*

$$U\otimes_{kH}V\cong\bigoplus_{\alpha\in\mathcal{I}_\theta}kG_{\gamma,\alpha}$$

as (kG,kG)-bimodules, where $\mathcal{I}_\theta=\big\{\alpha\in G^\natural=K^\natural\mid\operatorname{Res}^K_J\alpha=\theta\big\}$.

Proof: 1. Let X be a vertex of U. Then X is a diagonal subgroup of $P\times H$, so $|X|\le|P|$, and U is a direct summand of $\operatorname{Ind}_X^{G\times H}k$. Similarly, if Y is a vertex of V, then Y is a diagonal subgroup of $H\times P$, hence $|Y|\le|P|$, and V is a direct summand of $\operatorname{Ind}_Y^{H\times G}k$. It follows that $U\otimes_{kH}V$ is a direct summand of

$$\bigoplus_{h\in p_2(X)\backslash H/p_1(Y)}\operatorname{Ind}_{X*^{(h,1)}Y}^{G\times G}k.$$

So the vertices of the indecomposable summands of $U\otimes_{kH}V$ are contained (up to conjugation) in some group $X*{}^{(h,1)}Y$, which has order at most $\min(|X|,|Y|)$. If one of them has order $|P|$, then $|P|\le|X|\le|P|$, hence $|X|=|P|$, and $|P|\le|Y|\le|P|$, so $|Y|=|P|$.

By Lemma 4.5, it follows that there is an embedding $\sigma : H \hookrightarrow G$ with $P \le \sigma(H) \le G$, and $\lambda \in H^\natural$, such that $U \cong \mathrm{Ind}_{\Delta_\sigma(H)}^{G\times H} k_\lambda$. Similarly, swapping G and H, there is an embedding $\tau : H \to G$ with $P \le \tau(H) \le G$, and $\mu \in H^\natural$, such that $V \cong \mathrm{Ind}_{\Delta_\tau^o(H)}^{H\times G} k_\mu$, where $\Delta_\tau^o(H) = \Big\{\big(h, \tau(h)\big) \mid h \in H\Big\}$. Then

$$U \otimes_{kH} V \cong \mathrm{Ind}_{\Delta_\sigma(H)*\Delta_\tau^o(H)}^{G\times G}(k_\lambda \otimes_k k_\mu).$$

Now $\sigma(H)/P$ and $\tau(H)/P$ are subgroups of the same order of the cyclic group K, so $\sigma(H) = \tau(H)$. Set $I := \tau(H)$. There is a unique automorphism ψ of I such that $\psi\big(\tau(h)\big) = \sigma(h)$ for all $h \in H$. Then

$$\Delta_\sigma(H) * \Delta_\tau^o(H) = \Big\{\big(\sigma(h), \tau(h)\big) \mid h \in H\Big\} = \big\{(\psi(x), x) \mid x \in I\big\} = \Delta_\psi(I).$$

Moreover $k_\lambda \otimes k_\mu$ is one dimensional, so there is a unique $\zeta \in I^\natural$ such that $k_\lambda \otimes k_\mu \cong k_\zeta$ as kI-modules, defined by $\zeta(x) = (\lambda\mu)\big(\tau^{-1}(x)\big)$ for $x \in I$. This completes the proof of Assertion 1.

2. By Lemma 4.4, an essential diagonal p-permutation bimodule M is isomorphic to $kG_{\gamma,\lambda}$ for some $\gamma \in \mathrm{Aut}(G)$ and $\lambda \in G^\natural$. Then M has vertex $\Delta_\gamma(P)$, of order $|P|$, so the conclusion of Assertion 1 holds. Hence there is a subgroup I of G, containing P, an automorphism ψ of I, and $\zeta \in I^\natural$, such that

$$U \otimes_{kH} V \cong \mathrm{Ind}_{\Delta_\psi(I)}^{G\times G} k_\zeta.$$

In particular $\Delta_\gamma(P)$ is contained in $\Delta_\psi(P)$, up to conjugation, and we can assume that $\Delta_\gamma(P) = \Delta_\psi(P)$, i.e. that ψ is the restriction of γ to P. We have $\Delta_\psi(P) \le \Delta_\psi(I) \le N := N_{G\times G}\big(\Delta_\psi(P)\big) = N_{G\times G}\big(\Delta_\gamma(P)\big)$. So N fits in a short exact sequence of groups

$$1 \longrightarrow Z(P) \longrightarrow N \xrightarrow{p_2} G \longrightarrow 1.$$

Similarly, the group $\overline{N} := N/\Delta_\psi(P)$ fits in the sequence

$$1 \longrightarrow Z(P) \longrightarrow \overline{N} \longrightarrow K \longrightarrow 1.$$

This sequence splits via $x \in K \mapsto \big(\gamma(x), x\big)\Delta_\psi(P)$, and we view K as a subgroup of $\overline{N}$ via this map.

The group $\overline{\Delta}_\psi(I) = \Delta_\psi(I)/\Delta_\psi(P)$ is a subgroup of $\overline{N}$, and intersects $Z(P)$ trivially. So $\overline{\Delta}_\psi(I)$ is isomorphic to a subgroup J of K. Let $\theta : J \to k^\times$ be the image of $\overline{\zeta}$ under this isomorphism.

Since $\Delta_\psi(P)$ acts trivially on k_ζ, we have $k_\zeta = \mathrm{Inf}_{\overline{\Delta}_\psi(I)}^{\Delta_\psi(I)} k_{\overline{\zeta}}$, where $\overline{\zeta}$ is the homomorphism $\overline{\Delta}_\psi(I) \to k^\times$ corresponding to ζ. Hence we have

$$\begin{aligned}
U \otimes_{kH} V &\cong \mathrm{Ind}_{\Delta_\psi(I)}^{G\times G} k_\zeta \\
&= \mathrm{Ind}_N^{G\times G}\mathrm{Ind}_{\Delta_\psi(I)}^N \mathrm{Inf}_{\overline{\Delta}_\psi(I)}^{\Delta_\psi(I)} k_{\overline{\zeta}} \\
&\cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^N \mathrm{Ind}_{\overline{\Delta}_\psi(I)}^{\overline{N}} k_{\overline{\zeta}} \\
&\cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^N \mathrm{Ind}_{\overline{\Delta}_\psi(I)}^{\overline{N}} \mathrm{Iso}_J^{\overline{\Delta}_\psi(I)} k_\theta \\
&\cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^N \mathrm{Ind}_K^{\overline{N}} \mathrm{Ind}_J^K k_\theta .
\end{aligned}$$

Now K is cyclic, so $\mathrm{Ind}_J^K k_\theta = \sum_{\substack{\alpha\in K^\natural \\ \mathrm{Res}_J^K \alpha=\theta}} k_\alpha$, and if $\alpha \in K^\natural$, then $\mathrm{Ind}_K^{\overline{N}} k_\alpha$ is an indecomposable projective $k\overline{N}$-module. Then $\mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^N\mathrm{Ind}_K^{\overline{N}} k_\alpha \cong kG_{\gamma,\alpha}$. Now

$$U \otimes_{kH} V \cong \bigoplus_{\beta\in\mathcal{I}_\theta} kG_{\gamma,\beta},$$

as was to be shown. □

Notation 4.7: *Assume that 4.3 holds.*

1. *We abuse notation identifying $\lambda \in K^\natural$ with $k_\lambda \in R_k(K)$.*
2. *We set $\overline{R}_k(K) = R_k(K)/\sum_{L<K} \mathrm{Ind}_L^K R_k(L)$, and we let $\alpha \mapsto \overline{\alpha}$ denote the projection map.*
3. *Let $\gamma \in \mathrm{Aut}(G)$. Then $\overline{N}_{G\times G}\big(\Delta_\gamma(P)\big) \cong Z(P) \rtimes K$, so taking coinvariants by $Z(P)$ yields an isomorphism*

$$v \in \mathrm{Proj}\Big(k\overline{N}_{G\times G}\big(\Delta_\gamma(P)\big)\Big) \mapsto v_{Z(P)} \in R_k(K).$$

For $u \in T^\Delta(G,G)$, let $r_\gamma(u)$ denote $\overline{u[\Delta_\gamma(P)]_{Z(P)}} \in \overline{R}_k(K)$.

We note that $\sum_{L<K} \mathrm{Ind}_L^K R_k(L)$ is an ideal of the ring $R_k(K)$, so $\overline{R}_k(K)$ has a natural quotient ring structure. Moreover, the group $\mathrm{Aut}(G)$ acts on $G^\natural = K^\natural$, and $\mathrm{Inn}(G)$ acts trivially on $G^\natural$, since $[G,G] \le P$. So $\mathrm{Out}(G)$ acts on $R_k(K)$ and $\overline{R}_k(K)$ by ring automorphisms.

Notation 4.8: *We denote by* $\mathrm{Out}(G) \ltimes \overline{R}_k(K)$ *the semidirect product of* $\mathrm{Out}(G)$ *with* $\overline{R}_k(K)$*, i.e.*

$$\mathrm{Out}(G) \ltimes \overline{R}_k(K) = \bigoplus_{\gamma\in\mathrm{Out}(G)} \gamma \ltimes \overline{R}_k(K),$$

where $\gamma \ltimes \overline{R}_k(K)$ *denotes a copy of* $\overline{R}_k(K)$ *indexed by* γ.

Then $\mathrm{Out}(G) \ltimes \overline{R}_k(K)$ *is a ring for the product defined by*

$$\forall \gamma, \delta \in \mathrm{Out}(G),\ \forall \lambda, \mu \in R_k(K),\ (\gamma \ltimes \overline{\lambda}) \cdot (\delta \ltimes \overline{\mu}) := (\gamma \circ \delta) \ltimes \overline{\big((\lambda \circ \delta) \times \mu\big)}.$$

We set

$$\mathrm{Out}(G) \ltimes \mathsf{R}\overline{R}_k(K) := \mathsf{R} \otimes_{\mathbb{Z}} \big(\mathrm{Out}(G) \ltimes \overline{R}_k(K)\big) \cong \bigoplus_{\gamma \in \mathrm{Out}(G)} \gamma \ltimes \mathsf{R}\overline{R}_k(K).$$

In the next statement, we recall explicitly Hypothesis 4.3, for the reader's convenience.

Theorem 4.9: *Let* G *be a group of the form* $P \rtimes K$*, where* P *is a* p*-group and* K *is a cyclic* p'*-group of order invertible in* R*, acting faithfully on* P*. Then:*

1. *The map*

$$\gamma \ltimes \overline{\alpha} \in \mathrm{Out}(G) \ltimes \mathsf{R}\overline{R}_k(K) \mapsto \varepsilon_{\mathsf{R}}(kG_{\gamma,\alpha}) \in \mathcal{E}_{\mathsf{R}}(G),$$

   where $\alpha \in K^\natural$ *and* $\gamma \in \mathrm{Out}(G)$*, extends to a well defined algebra homomorphism* $\mathbf{T}$.

2. *The map*

$$\varepsilon_{\mathsf{R}}(u) \in \mathcal{E}_{\mathsf{R}}(G) \mapsto \sum_{\gamma \in \mathrm{Out}(G)} \gamma \ltimes r_\gamma(u) \in \mathrm{Out}(G) \ltimes \mathsf{R}\overline{R}_k(K),$$

   where $u \in \mathsf{R}T^\Delta(G,G)$*, is a well defined algebra homomorphism* $\mathbf{S}$.

3. *The maps*

$$\mathcal{E}_{\mathsf{R}}(G) \underset{\mathbf{T}}{\overset{\mathbf{S}}{\rightleftarrows}} \mathrm{Out}(G) \ltimes \mathsf{R}\overline{R}_k(K)$$

   are isomorphisms of algebras, inverse to each other.

Proof: Proving that the map $\mathbf{T}$ is well defined amounts to proving that if $u \in K^\natural$ is induced from a proper subgroup J of K, and if $\gamma \in \mathrm{Out}(G)$, then $\mathbf{T}(\gamma \ltimes \overline{u}) = 0$. Let $J < K$, and $\theta \in J^\natural$. Then $u = \mathrm{Ind}_J^K \theta = \sum_{\alpha \in \mathcal{I}_\theta} k_\alpha$, so

$$\mathbf{T}(\gamma \ltimes u) = \mathcal{E}_{\mathsf{R}}\Big(\bigoplus_{\alpha \in \mathcal{I}_\theta} kG_{\gamma,\alpha}\Big).$$

But setting $N := N_{G\times G}\big(\Delta_\gamma(P)\big)$ and $\overline{N} := N/\Delta_\gamma(P)$, we have

$$\begin{aligned}
\bigoplus_{\alpha\in\mathcal{I}_\theta} kG_{\gamma,\alpha} &\cong \bigoplus_{\alpha\in\mathcal{I}_\theta} \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N}\mathrm{Ind}_K^{\overline{N}}k_\alpha\\
&\cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N}\mathrm{Ind}_K^{\overline{N}}\Big(\bigoplus_{\alpha\in\mathcal{I}_\theta} k_\alpha\Big)\\
&\cong \mathrm{Ind}_N^{G\times G}\mathrm{Inf}_{\overline{N}}^{N}\mathrm{Ind}_K^{\overline{N}}\mathrm{Ind}_J^K k_\theta\\
&\cong \mathrm{Ind}_N^{G\times G}\mathrm{Ind}_{\Delta_\gamma(P)\cdot\Delta_\gamma(K)}^{N}\mathrm{Inf}_K^{\Delta_\gamma(P)\cdot\Delta_\gamma(K)}\mathrm{Ind}_J^K k_\theta\\
&\cong \mathrm{Ind}_{\Delta_\gamma(G)}^{G\times G}\mathrm{Inf}_K^{\Delta_\gamma(G)}\mathrm{Ind}_J^K k_\theta\\
&\cong \mathrm{Ind}_{\Delta_\gamma(G)}^{G\times G}\mathrm{Ind}_{\Delta_\gamma(P)\cdot\Delta_\gamma(J)}^{\Delta_\gamma(G)}\mathrm{Inf}_J^{\Delta_\gamma(P)\cdot\Delta_\gamma(J)}k_\theta\\
&\cong \mathrm{Ind}_{\Delta_\gamma(P\cdot J)}^{G\times G}\mathrm{Inf}_J^{\Delta_\gamma(P\cdot J)}k_\theta.
\end{aligned}$$

But $p_2\big(\Delta_\gamma(P\cdot J)\big) = P\cdot J < G$ since $J < K$. Hence $\mathcal{E}_{\mathsf{R}}\Big(\bigoplus_{\alpha\in\mathcal{I}_\theta} kG_{\gamma,\alpha}\Big) = 0$, as was to be shown, so the map $\mathbf{T}$ is well defined.

Now comparing the products in Lemma 4.2 and Notation 4.8, we get that $\mathbf{T}$ is a homomorphism of R-algebras. Moreover the identity element of $\mathrm{Out}(G)\ltimes \mathsf{R}\overline{R}_k(K)$, which is $\mathrm{Id}\ltimes\overline{1}$, is mapped by $\mathbf{T}$ to $\mathcal{E}_{\mathsf{R}}(kG_{\mathrm{Id},1}) = \mathcal{E}_{\mathsf{R}}(kG)$, which is the identity element of $\mathcal{E}_{\mathsf{R}}(G)$.

2. Proving that the map $\mathbf{S}$ is well defined amounts to proving that if H is a finite group with $|H| < |G|$, if U (resp. V) is a right (resp. left) essential diagonal p-permutation (kG, kH)-bimodule (resp. (kH, kG)-bimodule), then $\mathbf{S}(U\otimes_{kH} V) = 0$. So let $\gamma\in\mathrm{Aut}(G)$ such that $r_\gamma(U\otimes_{kH} V)\neq 0$. Then in particular $(U\otimes_{kH} V)[\Delta_\gamma(P)]\neq 0$, so $U\otimes_{kH} V$ admits an indecomposable summand with vertex $\Delta_\gamma(P)$. By Lemma 4.4, this summand is isomorphic to $kG_{\gamma,\lambda}$, for some $\lambda\in G^\natural$. By Theorem 4.6, there is a subgroup J of K with $P\cdot J\cong H$, and $\theta\in J^\natural$ such that

$$U\otimes_{kH} V\cong \bigoplus_{\alpha\in\mathcal{I}_\theta} kG_{\gamma,\alpha}.$$

Now by (4.4.1)

$$(U\otimes_{kH} V)[\Delta_\gamma(P)]\cong \bigoplus_{\alpha\in\mathcal{I}_\theta}\mathrm{Ind}_K^{\overline{N}}k_\alpha \cong \mathrm{Ind}_K^{\overline{N}}\big(\bigoplus_{\alpha\in I_\theta} k_\alpha\big)\cong kZ(P)\otimes_k \mathrm{Ind}_J^K k_\theta.$$

It follows that in $\overline{R}_k(K)$, we have $r_\gamma(U\otimes_{kH} V) = \overline{\mathrm{Ind}_J^K k_\theta} = 0$ since $J < K$ as $P\cdot J\cong H$ and $|H| < |G| = |P||K|$. This contradiction shows that S is well defined.

We postpone the proof that $\mathbf{S}$ is an algebra homomorphism at the end of the proof of Assertion 3.

3. Let $\gamma \in \mathrm{Aut}(G)$ and $\alpha \in K^\natural$. Then $kG_{\gamma,\alpha}$ has vertex $\Delta_\gamma(P)$, and $kG_{\gamma,\alpha}[\Delta_\gamma(P)] \cong kZ(P) \otimes k_\alpha$ by (4.4.1). We also get from Lemma 4.4 that $kG_{\gamma,\alpha}[\Delta_\delta(P)] = 0$ if $\delta \in \mathrm{Aut}(G)$ and $\delta \neq \gamma$ in $\mathrm{Out}(G)$: Indeed, if $\Delta_\gamma(P)$ and $\Delta_\delta(P)$ are conjugate in $G \times G$, then the restriction of δ to P is equal to the restriction of δ to P, up to an automorphism given by conjugation by some element of G, which we may assume to be trivial. Then $\delta^{-1}\gamma$ is inner, by Lemma 3.1.

It follows that $\mathbf{S}\big(\mathcal{E}_{\mathsf{R}}(kG_{\gamma,\alpha})\big) = \gamma \ltimes \overline{\alpha}$. Since $\mathbf{T}(\gamma \ltimes \overline{\alpha}) = \mathcal{E}_{\mathsf{R}}(kG_{\gamma,\alpha})$, the maps $\mathbf{S}$ and $\mathbf{T}$ are inverse to each other. In particular, they are bijections, so $\mathbf{S}$ is a map of R-algebras, as $\mathbf{T}$ is. This completes the proof. □

5 The simple functors

5.1. We want to consider the simple diagonal p-permutation functors, so by general arguments, we can assume that our ring R of coefficients is a field $\mathbb{F}$. Moreover, in order to apply the results of the previous sections, we want that p'-groups have order invertible in $\mathbb{F}$. So we are left with the cases where $\mathbb{F}$ has characteristic 0 or p.

If S is a simple diagonal p-permutation functor over $\mathbb{F}$, and H is a finite group of minimal order such that $S(H) \neq 0$, then $V = S(H)$ is a simple module for the essential algebra $\mathcal{E}_{\mathbb{F}}(H)$. In particular $\mathcal{E}_{\mathbb{F}}(H) \neq 0$, so $H = L\langle u\rangle$ for some D^Δ-pair (L, u). Moreover, we have an isomorphism of algebras

$$\mathcal{E}_{\mathbb{F}}\big(L\langle u\rangle\big) \cong \mathrm{Out}\big(L\langle u\rangle\big) \ltimes \mathbb{F}\overline{R}_k(L\langle u\rangle).$$

5.2. Conversely, if (L, u) is a D^Δ-pair, and V is a simple $\mathcal{E}_{\mathbb{F}}\big(L\langle u\rangle\big)$-module, then we denote by $S_{L\langle u\rangle,V}$ the unique simple diagonal p-permutation functor with minimal group $L\langle u\rangle$ and such that $S_{L\langle u\rangle,V}\big(L\langle u\rangle\big) \cong V$ as an $\mathcal{E}_{\mathbb{F}}\big(L\langle u\rangle\big)$-module. The evaluation of $S_{L\langle u\rangle,V}$ at a finite group G is isomorphic to

$$S_{L\langle u\rangle,V}(G) \cong \Big(\mathbb{F}T^\Delta\big(G, L\langle u\rangle\big) \otimes_{\mathcal{E}_{\mathbb{F}}(L\langle u\rangle)} V\Big)/\mathcal{R},$$

where $\mathcal{R}$ is the subspace generated by all finite sums $\sum_{i\in I} f_i \otimes v_i$, with $f_i \in \mathbb{F}T^\Delta\big(G, L\langle u\rangle\big)$ and $v_i \in V$ for $i \in I$, such that $\sum_{i\in I} \pi(\varphi \circ f_i)\cdot v_i = 0$ for all $\varphi \in \mathbb{F}T^\Delta(L\langle u\rangle, G)$, where $\pi : \mathbb{F}T^\Delta\big(L\langle u\rangle, L\langle u\rangle\big) \to \mathcal{E}_{\mathbb{F}}(L\langle u\rangle)$ is the projection map.

5.3. In particular, let $f \in \mathbb{F}T^\Delta\big(G, L\langle u\rangle\big)$ be of the form $f = \mathrm{Ind}_N^{G\times L\langle u\rangle}\mathrm{Inf}_{\overline{N}}^N E$, where

- $N = N_{G\times L\langle u\rangle}\big(\Delta(P,\gamma,R)\big)$, for some $R \leq L$ and $\gamma : R \xrightarrow{\cong} P \leq G$,
- $\overline{N} = N/\Delta(P,\gamma,R)$,
- E is a projective $k\overline{N}$-module.

If there exists some $v \in V$ such that $f \otimes v \notin \mathcal{R}$, then there exists $\varphi \in \mathbb{F}T^{\Delta}(L\langle u\rangle, G)$ such that $\pi(\varphi \circ f) \neq 0$. In particular, the Brauer quotient $(\varphi \circ f)[\Delta(L, \theta, L)]$ has to be non zero for some $\theta \in \mathrm{Aut}(L)$, which forces $R = L$.

5.4. Moreover if $f \in \mathbb{F}T^{\Delta}\big(G, L\langle u\rangle\big)$ can be factorized through a group of order strictly smaller than the order of $L\langle u\rangle$, then $f \otimes v \in \mathcal{R}$ for any $v \in V$. So if there exists $v \in V$ with $f \otimes v \notin \mathcal{R}$, then in particular $p_2(N) = L\langle u\rangle$.

Remark 5.5: This condition $p_2(N) = L\langle u\rangle$ means that for any $x \in L\langle u\rangle$, there exists $g \in N_G(P)$ such that ${}^g\gamma(l) = \gamma({}^x l)$ for all $l \in L$. Equivalently, there exists $s \in N_G(P)$ such that ${}^s\gamma(l) = \gamma({}^u l)$ for all $l \in L$: Suppose indeed that such an element s exists. Any element $x \in L$ is equal to $l_0 u^\alpha$ for some $l_0 \in L$ and $\alpha \in \mathbb{N}$. Then

$$\gamma \circ i_x = \gamma \circ i_{l_0} \circ i_{u^\alpha} = i_{\gamma(l_0)} \circ \gamma \circ (i_u)^\alpha = i_{\gamma(l_0)} \circ (i_s)^\alpha \circ \gamma = i_{\gamma(l_0)s^\alpha} \circ \gamma,$$

that is $i_g \circ \gamma = \gamma \circ i_x$, for $g = \gamma(l_0)s^\alpha$.

In other words, saying that $p_2(N) = L\langle u\rangle$ amounts to saying that (P, γ) belongs to the set

$$\mathcal{P}(G, L, u) := \big\{(P, \gamma) \mid \gamma : L \overset{\cong}{\to} P \leq G,\ \exists s \in N_G(P),\ i_s \circ \gamma = \gamma \circ i_u\big\}.$$

For $\varphi \in \mathrm{Aut}\big(L\langle u\rangle\big)$, we have that

$$\begin{aligned} N_{G\times L\langle u\rangle}\big(\Delta(P, \gamma\varphi, L)\big) &= \big\{(a, b) \mid (a, \varphi(b)) \in N_{G\times L\langle u\rangle}\big(\Delta(P, \gamma, L)\big)\big\} \\ &= (1 \times \varphi^{-1})N_{G\times L\langle u\rangle}\big(\Delta(P, \gamma, L)\big) \end{aligned}$$

so the set $\mathcal{P}(G, L, u)$ is a $\Big(G, \mathrm{Aut}\big(L\langle u\rangle\big)\Big)$-biset by

$$\forall g \in G,\ \forall \varphi \in \mathrm{Aut}(L\langle u\rangle),\ g \cdot (P, \gamma) \cdot \varphi = ({}^gP, i_g\gamma\varphi),$$

5.6. Let $\mathcal{T}(G, L, u)$ denote the set of triples (P, γ, E), where $(P, \gamma) \in \mathcal{P}(G, L, u)$ and E is an indecomposable projective $kN\big(\Delta(P, \gamma, L)\big)/\Delta(P, \gamma, L)$-module. The set $\mathcal{T}(G, L, u)$ is also a $\Big(G, \mathrm{Aut}\big(L\langle u\rangle\big)\Big)$-biset by

$$\forall g \in G,\ \forall \varphi \in \mathrm{Aut}(L\langle u\rangle),\ g \cdot (P, \gamma, E) \cdot \varphi = ({}^gP, i_g\gamma\varphi, {}^gE^\varphi)$$

where ${}^gE^\varphi$ is the $kN\big(\Delta({}^gP, i_g\gamma\varphi, L)\big)/\Delta({}^gP, i_g\gamma\varphi, L)$-module obtained from E via the group isomorphism

$$(a, b)\Delta({}^gP, i_g\gamma\varphi, L) \mapsto \big(a^g, \varphi(b)\big)\Delta(P, \gamma, L)$$

from $kN\big(\Delta({}^gP, i_g\gamma\varphi, L)\big)/\Delta({}^gP, i_g\gamma\varphi, L)$ to $kN\big(\Delta(P, \gamma, L)\big)/\Delta(P, \gamma, L)$.

For $(P,\gamma,E)\in\mathcal{T}(G,L,u)$, set

$$T(P,\gamma,E)=\mathrm{Ind}_N^{G\times L\langle u\rangle}\mathrm{Inf}_{\overline{N}}^{N}E,$$

where $N=N_{G\times L\langle u\rangle}\big(\Delta(P,\gamma,L)\big)$ and $\overline{N}=N/\Delta(P,\gamma,L)$. Then $T(P,\gamma,E)$ is a diagonal p-permutation $(kG,kL\langle u\rangle)$-bimodule, and we abuse notation writing $T(P,\gamma,E)\in\mathbb{F}T^{\Delta}(G,L\langle u\rangle)$. We observe that for $(g,\varphi)\in G\times\mathrm{Aut}(L\langle u\rangle)$, we have

$$T({}^gP,i_g\gamma\varphi,{}^gE^{\varphi})\cong T(P,\gamma,E)\otimes_{L\langle u\rangle}k\big(L\langle u\rangle\big)_{\varphi},$$

where $k\big(L\langle u\rangle\big)_{\varphi}$ is the bimodule $k\big(L\langle u\rangle\big)$ twisted by φ on the right, that is $x\cdot m\cdot y$ (in $k\big(L\langle u\rangle\big)_{\varphi}$) $=xm\varphi(y)$ (in $k\big(L\langle u\rangle\big)$).

5.7. Now a lemma:

Lemma 5.8: *Let J be a finite group, let $K\trianglelefteq J$ such that J/K is a cyclic p'-group. Let E be an indecomposable projective kJ-module, let V be an indecomposable summand of Res_K^JE, and H be the inertia group of V in J. Then V extends to an indecomposable projective kH-module F, and there exists a group homomorphism $\lambda:H/K\to k^{\times}$ such that*

$$E\cong\mathrm{Ind}_H^J(F\otimes_k\mathrm{Inf}_{H/K}^{H}k_{\lambda}).$$

Proof: Use Theorem 3.13.2 in [1], and Theorem 4.1 of [7]. □

5.9. Let $(P,\gamma,E)\in\mathcal{T}(G,L,u)$. Set $N_{P,\gamma}=N_{G\times L\langle u\rangle}\big(\Delta(P,\gamma,L)\big)$, and $\overline{N}_{P,\gamma}=N_{P,\gamma}/\Delta(P,\gamma,L)$, so E is an indecomposable projective $k\overline{N}_{P,\gamma}$-module. There is an exact sequence of groups

$$1\to C_G(P)\to\overline{N}_{P,\gamma}\to\langle u\rangle\to 1,\tag{5.9.1}$$

so by the previous lemma, if V is an indecomposable summand of $\mathrm{Res}_{C_G(P)}^{\overline{N}_{P,\gamma}}E$, and H its stabilizer in $\overline{N}_{P,\gamma}$, there exists an indecomposable projective kH-module F such that

$$E\cong\mathrm{Ind}_H^{\overline{N}_{P,\gamma}}F.$$

From this follows that

$$\begin{aligned}T(P,\gamma,E)&=\mathrm{Ind}_{N_{P,\gamma}}^{G\times L\langle u\rangle}\mathrm{Inf}_{\overline{N}_{P,\gamma}}^{N_{P,\gamma}}E\\&\cong\mathrm{Ind}_{N_{P,\gamma}}^{G\times L\langle u\rangle}\mathrm{Inf}_{\overline{N}_{P,\gamma}}^{N_{P,\gamma}}\mathrm{Ind}_H^{\overline{N}_{P,\gamma}}F\\&\cong\mathrm{Ind}_{N_{P,\gamma}}^{G\times L\langle u\rangle}\mathrm{Ind}_{\hat{H}}^{N_{P,\gamma}}\mathrm{Inf}_H^{\hat{H}}F\cong\mathrm{Ind}_{\hat{H}}^{G\times L\langle u\rangle}\mathrm{Inf}_H^{\hat{H}}F,\end{aligned}$$

where $\hat{H}$ is the inverse image in $N_{P,\gamma}$ of $H\le\overline{N}_{P,\gamma}$ by the projection map $N_{P,\gamma}\to\overline{N}_{P,\gamma}$. Now $T(P,\gamma,E)$ factors through the second projection of $\hat{H}\le G\times L\langle u\rangle$, so we can assume that this projection is the whole of $L\langle u\rangle$, i.e. equivalently that $H = \overline{N}_{P,\gamma}$. In other words, by Theorem 4.1 of [7] already quoted above, we can assume that $\mathrm{Res}^{\overline{N}_{P,\gamma}}_{C_G(P)} E$ is indecomposable. We denote by $\mathrm{Pim}^\sharp(k\overline{N}_{P,\gamma})$ the set of isomorphism classes of such indecomposable projective $k\overline{N}_{P,\gamma}$-modules, and by $\mathcal{T}^\sharp(G, L, u)$ the subset of $\mathcal{T}(G, L, u)$ consisting of triples (P, γ, E) such that $E \in \mathrm{Pim}^\sharp(k\overline{N}_{P,\gamma})$.

5.10. It follows from the above remarks that $S_{L\langle u\rangle,V}(G)$ is generated by the images of the elements $T(P, \gamma, E) \otimes v$, where

- (P, γ) runs through a set $[\mathcal{P}(G, L, u)]$ of representatives of orbits of $\big(G \times \mathrm{Aut}(L\langle u\rangle)\big)$ on $\mathcal{P}(G, L, u)$,
- $E \in \mathrm{Pim}^\sharp(k\overline{N}_{P,\gamma})$,
- $v \in V$.

In other words, we have a surjective map

$$\bigoplus_{\substack{(P,\gamma)\in[\mathcal{P}(G,L,u)]\\ E\in\mathrm{Pim}^\sharp(k\overline{N}_{P,\gamma})}} T(P, \gamma, E) \otimes V \longrightarrow S_{L\langle u\rangle,V}(G)$$

sending $T(P, \gamma, E) \otimes v \in \mathbb{F}T^\Delta\big(G, L\langle u\rangle\big) \otimes V$ to its image in $S_{L\langle u\rangle,V}(G)$. The kernel of this map is the set of sums

$$\sum_{\substack{(P,\gamma)\in[\mathcal{P}(G,L,u)]\\ E\in\mathrm{Pim}^\sharp(k\overline{N}_{P,\gamma})}} T(P, \gamma, E) \otimes v_{P,\gamma,E}$$

where $v_{P,\gamma,E} \in V$, such that for any (Q, δ) in $[\mathcal{P}(G, L, u)]$ and any indecomposable projective $k\overline{N}_{Q,\delta}$-module F, or equivalently for any F in $\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})$

$$\sum_{\substack{(P,\gamma)\in[\mathcal{P}(G,L,u)]\\ E\in\mathrm{Pim}^\sharp(k\overline{N}_{P,\gamma})}} \pi\big(T^o(Q, \delta, F) \otimes_{kG} T(P, \gamma, E)\big) \cdot v_{P,\gamma,E} = 0, \tag{5.10.1}$$

where $T^o(Q, \delta, F)$ is the $(kL\langle u\rangle, kG)$-bimodule "opposite" of the $(kG, kL\langle u\rangle)$-bimodule $T(Q, \delta, F)$. In other words

$$T^o(Q, \delta, F) = \mathrm{Ind}^{L\langle u\rangle\times G}_{N^o_{Q,\delta}} \mathrm{Inf}^{N^o_{Q,\delta}}_{\overline{N}^o_{Q,\delta}} F^o$$

where

- $N^o_{Q,\delta} = N_{L\langle u\rangle\times G}\big(\Delta(L, \delta^{-1}, Q)\big)$,
- $\overline{N}^o_{Q,\delta} = N^o_{Q,\delta}/\Delta(L, \delta^{-1}, Q)$,

- F^o is the opposite module of F.

Now if $\pi\big(T^o(Q,\delta,F)\otimes_{kG}T(P,\gamma,E)\big)\neq 0$, there exists an automorphism θ of $L\langle u\rangle$ such that

$$\big(T^o(Q,\delta,F)\otimes_{kG}T(P,\gamma,E)\big)[\Delta(L,\theta,L)]\neq 0.$$

Hence there exist $V\leq G$, $\alpha: V\overset{\cong}{\to} L$, and $\beta: L\overset{\cong}{\to} V$ such that $\theta=\alpha\beta$ and

$$T^o(Q,\delta,F)[\Delta(L,\alpha,V)]\neq 0 \text{ and } T(P,\gamma,E)[\Delta(V,\beta,L)]\neq 0.$$

So $\Delta(Q,\delta,L)$ is conjugate to $\Delta(V,\alpha^{-1},L)$ in $G\times L\langle u\rangle$, and $\Delta(P,\gamma,L)$ is conjugate to $\Delta(V,\beta,L)$ in $G\times L\langle u\rangle$. By Remark 5.5, this amounts to saying that there exist $g,h\in G$ such that $(V,\alpha^{-1})=({}^gQ,i_g\delta)$ and $(V,\beta)=({}^hP,i_h\gamma)$. Hence $P=V^h={}^{h^{-1}g}Q$ and $\theta=\delta^{-1}i_{g^{-1}}\gamma$.

In other words, setting $z=h^{-1}g$, we have $P={}^zQ$ and $\gamma=i_z\delta\theta$, that is $(P,\gamma)=z\cdot(Q,\delta)\cdot\theta$. As (P,γ) and (Q,δ) are both in our set $[\mathcal{P}(G,L,u)]$ of representatives of $G\times\mathrm{Aut}(L\langle u\rangle)$-orbits on $\mathcal{P}(G,L,u)$, this forces $P=Q$ and $\gamma=\delta$.

Now Equation 5.10.1 reduces to the fact that for every (Q,δ) in $[\mathcal{P}(G,L,u)]$ and any $F\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})$

$$\sum_{E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})}\pi\big(T^o(Q,\delta,F)\otimes_{kG}T(Q,\delta,E)\big)\cdot v_{Q,\delta,E}=0. \tag{5.10.2}$$

It follows that

$$S_{L\langle u\rangle,V}(G)\cong\bigoplus_{(Q,\delta)\in[\mathcal{P}(G,L,u)]}\Big(\big(\underset{E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})}{\oplus}T(Q,\delta,E)\otimes V\big)/\mathcal{R}_{Q,\delta}\Big) \tag{5.10.3}$$

where the relations $\mathcal{R}_{Q,\delta}$ are given by (5.10.2) for every $F\in\mathrm{Pim}(k\overline{N}_{Q,\delta})$.

Now we set

$$G_{Q,\delta}=\{g\in G\mid \exists x\in L\langle u\rangle,\ (g,x)\in N_{G\times L\langle u\rangle}\big(\Delta(Q,\delta,L)\big)\}.$$

With this notation, the $\big(L\langle u\rangle,L\langle u\rangle\big)$-bimodule $M=T^o(Q,\delta,F)\otimes_{kG}T(Q,\delta,E)$ is isomorphic to

$$\begin{aligned} M &\cong \bigoplus_{g\in G_{Q,\delta}\backslash G/G_{Q,\delta}}\mathrm{Ind}_{N^o_{Q,\delta}*^{(g,1)}N_{Q,\delta}}^{L\langle u\rangle\times L\langle u\rangle}\big(\mathrm{Inf}_{\overline{N}^o_{Q,\delta}}^{N^o_{Q,\delta}}F^o\underset{kC_G(Q,{}^gQ)}{\otimes}{}^{(g,1)}\mathrm{Inf}_{\overline{N}_{Q,\delta}}^{N_{Q,\delta}}E\big)\\ &= \bigoplus_{g\in G_{Q,\delta}\backslash G/G_{Q,\delta}}\mathrm{Ind}_{N^o_{Q,\delta}*N_{{}^gQ,i_g\delta}}^{L\langle u\rangle\times L\langle u\rangle}\big(\mathrm{Inf}_{\overline{N}^o_{Q,\delta}}^{N^o_{Q,\delta}}F^o\underset{kC_G(Q,{}^gQ)}{\otimes}\mathrm{Inf}_{\overline{N}_{{}^gQ,i_g\delta}}^{N_{{}^gQ,i_g\delta}}{}^{(g,1)}E\big). \end{aligned} \tag{5.10.4}$$

5.11. Now another lemma:

Lemma 5.12: *Let G, H, K be finite groups, let $Z \leq X \leq G \times H$ and $T \leq Y \leq H \times K$ be subgroups. Set $D = k_2(X) \cap k_1(Y)$. Then $X/Z \underset{D}{\times} Y/T$ is a $X * Y$-set, and for $(u,v)Z \in X/Z$ and $(w,r)T \in Y/T$, the stabilizer of $(u,v)Z \underset{D}{\times} (w,r)T$ in $X * Y$ is equal to ${}^{(u,v)}Z * {}^{(w,r)}T$.*

Proof: This is straightforward. □

5.13. Let M_g denote the term of the direct sum (5.10.4) indexed by $g \in G$, that is
$$M_g = \mathrm{Ind}_{N^o_{Q,\delta} * N_{{}^gQ, i_g\delta}}^{L\langle u\rangle \times L\langle u\rangle} \big(\mathrm{Inf}_{\overline{N}^o_{Q,\delta}}^{N^o_{Q,\delta}} F^o \underset{kC_G(Q,{}^gQ)}{\otimes} \mathrm{Inf}_{\overline{N}_{{}^gQ,i_g\delta}}^{N_{{}^gQ,i_g\delta}\,(g,1)} E\big).$$
To apply the previous lemma, we set
$$X = N^o_{Q,\delta},\ Y = N_{{}^gQ, i_g\delta},\ Z = \Delta^o(Q, \delta, L),\ T = \Delta({}^gQ, i_g\delta, L).$$
Then $k_2(X) = C_G(Q)$ and $k_1(Y) = C_G({}^gQ)$, thus $D = C_G(Q, {}^g Q)$. Moreover $Z \trianglelefteq X$ and $T \trianglelefteq Y$

Now since E is a direct summand of $k\overline{N}_{Q,\delta}$, it follows that $\mathrm{Inf}_{\overline{N}_{{}^gQ,i_g\delta}}^{N_{{}^gQ,i_g\delta}\,(g,1)} E$ is a direct summand of kY/T. Similarly $\mathrm{Inf}_{\overline{N}^o_{Q,\delta}}^{N^o_{Q,\delta}} F^o$ is a direct summand of kX/Z. Hence $\mathrm{Inf}_{\overline{N}^o_{Q,\delta}}^{N^o_{Q,\delta}} F^o \otimes_{kC_G(Q,{}^gQ)} \mathrm{Inf}_{\overline{N}_{{}^gQ,i_g\delta}}^{N_{{}^gQ,i_g\delta}\,(g,1)} E$ is a direct summand of
$$kX/Z \otimes_{kC_G(Q,{}^gQ)} kY/T \cong k(X/Z) \underset{D}{\times} (Y/T).$$
By the previous lemma, for $(u,v)Z \in X/Z$ and $(w,r)T \in Y/T$, the stabilizer of $(u,v)Z \underset{D}{\times} (w,r)T$ in $X*Y$ is equal to ${}^{(u,v)}Z * {}^{(w,r)}T = Z*T$. Hence the vertices of the indecomposable direct summands of $kX/Z \otimes_{kC_G(Q,{}^gQ)} kY/T$ are subgroups of $Z * T$.

It follows that if $\pi(M_g) \neq 0$, then there exists $\theta \in \mathrm{Aut}(L\langle u\rangle)$ such that $\Delta(L, \theta, L)$ is conjugate in $L\langle u\rangle \times L\langle u\rangle$ to a subgroup of $Z * T$. Up to changing θ by some inner automorphism of $L\langle u\rangle$, we can assume that $\Delta(L, \theta, L) \leq Z * T$. But
$$\begin{aligned} Z * T &= \Delta^o(Q, \delta, L) * \Delta({}^gQ, i_g\delta, L) \\ &= \Big\{(a,b) \in L\langle u\rangle \times L\langle u\rangle \mid \exists c \in G,\ \Big\{\begin{matrix}(c,a) \in \Delta(Q,\delta,L) \\ (c,b) \in \Delta({}^gQ, i_g\delta, L)\end{matrix}\Big\} \\ &= \big\{\big(\delta^{-1}(c), \delta^{-1}(c^g)\big) \mid c \in Q \cap {}^gQ\big\}. \end{aligned}$$
Then $Z * T$ contains $\Delta(L, \theta, L)$ if and only if $Q = {}^qQ$ and $\theta(l) = \delta^{-1}({}^g\delta(l))$ for any $l \in L$. In other words $g \in N_G(Q)$ and $\delta^{-1}i_g\delta$ is the restriction of θ to L.

For $g \in N_G(Q)$, we set ${i_g}^\delta = \delta^{-1}i_g\delta \in \mathrm{Aut}(L)$ and
$$\widehat{G}_{Q,\delta} = \{g \in N_G(Q) \mid \exists \theta \in \mathrm{Aut}(L\langle u\rangle),\ {i_g}^\delta = \theta_{|L}\}.$$

With this notation, we see that $\pi(M_g) = 0$ unless $g \in \widehat{G}_{Q,\delta}$.

We also observe that $G_{Q,\delta}$ is a normal subgroup of $\widehat{G}_{Q,\delta}$, and we set

$$\overline{G}_{Q,\delta} = \widehat{G}_{Q,\delta}/G_{Q,\delta}.$$

We observe moreover that for $g \in \widehat{G}_{G,\delta}$, there is a unique $\theta \in \mathrm{Aut}(L\langle u\rangle)$ such that $\theta_{|L} = {i_g}^{\delta}$, up to inner automorphism, thanks to Lemma 3.1.

5.14. For $g \in \widehat{G}_{Q,\delta}$, we will denote by $\theta_g \in \mathrm{Out}(L\langle u\rangle)$ the unique outer automorphism such that $(\theta_g)_{|L} = {i_g}^{\delta}$, and by $\hat{\theta}_g$ a representative of θ_g in $\mathrm{Aut}(L\langle u\rangle)$. The map $g \in \widehat{G}_{Q,\delta} \mapsto \theta_g \in \mathrm{Out}(L\langle u\rangle)$ is a group homomorphism, with kernel $G_{Q,\delta}$. In other words $\overline{G}_{Q,\delta} = \widehat{G}_{Q,\delta}/G_{Q,\delta}$ embeds in $\mathrm{Out}(L\langle u\rangle)$.

Now let $(a,b) \in N_{Q,\delta}$, i.e. $(a,b) \in G \times L\langle u\rangle$, and ${}^a\delta(l) = \delta({}^bl)$ for all $l \in L$. If $g \in \widehat{G}_{Q,\delta}$ we claim that $\big({}^ga, \hat{\theta}_g(b)\big)$ also lies in $N_{Q,\delta}$. Indeed for $l \in L$, setting $l' = \hat{\theta}_g^{-1}(l)$, we have $\delta(l) = i_g\delta(l')$, so

$$\begin{aligned} {}^{({}^ga)}\delta(l) &= {}^{({}^ga)}i_g\delta(l) = ga\delta(l')a^{-1}g^{-1} \\ &= g\delta({}^bl')g^{-1} = i_g\delta({}^bl'), \end{aligned}$$

whereas

$$\begin{aligned} \delta({}^{\hat{\theta}_g(b)}l) &= \delta\big({}^{\hat{\theta}_g(b)}\hat{\theta}_g(l')\big) \\ &= \delta\big(\hat{\theta}_g({}^bl')\big) \\ &= \delta\delta^{-1}i_g\delta({}^bl') = i_g\delta({}^bl'), \end{aligned}$$

proving the claim.

In other words, the map $\Phi_g : (a,b) \mapsto \big({}^ga, \hat{\theta}_g(b)\big)$ is an automorphism of $N_{Q,\delta}$. Moreover, it sends $\Delta(Q,\delta,L)$ to itself. Indeed, for $l \in L$, we have

$$\big({}^g\delta(l), \hat{\theta}_g(l)\big) = \big(i_g\delta(l), \delta^{-1}i_g\delta(l)\big) = \Big(\delta\big(\theta_g(l)\big), \theta_g(l)\Big).$$

It follows that Φ_g induces an automorphism $\bar{\Phi}_g$ of $\overline{N}_{Q,\delta}$. But there is a little more: Let $\overline{N}^\flat_{Q,\delta}$ denote the quotient $\overline{N}_{Q,\delta}/Z(Q)$. Then by the above lemma, for $g,h \in \widehat{G}_{Q,\delta}$, the composition $\Phi_g\Phi_h$ is equal to Φ_{gh} modulo an inner automorphism induced by an element of $Z(L)$. It follows that Φ_g induces an automorphism $\Phi^\flat_g$ of $\overline{N}^\flat_{Q,\delta}$, and that $\Phi^\flat_g\Phi^\flat_h = \Phi^\flat_{gh}$. In other words $\widehat{G}_{Q,\delta}$ acts on $\overline{N}^\flat_{Q,\delta}$.

5.15. Assuming now that $g \in \widehat{G}_{Q,\delta}$, we have

$$M_g = \mathrm{Ind}_{N^o_{Q,\delta} * N_{Q,i_g\delta}}^{L\langle u\rangle \times L\langle u\rangle}\big(\mathrm{Inf}_{\overline{N}^o_{Q,\delta}}^{N^o_{Q,\delta}} F^o \underset{kC_G(Q)}{\otimes} \mathrm{Inf}_{\overline{N}_{Q,i_g\delta}}^{N_{Q,i_g\delta}} {}^{(g,1)}E\big).$$

Moreover

$$\begin{aligned} N^o_{Q,\delta} * N_{Q,i_g\delta} &= \{(a,b) \in L\langle u\rangle \times L\langle u\rangle \mid \exists c \in G, (c,a) \in N_{Q,\delta}, (c,b) \in N_{Q,i_g\delta}\} \\ &= \left\{ (a,b) \in L\langle u\rangle \times L\langle u\rangle \mid \exists c \in G, \forall l \in L, \begin{cases} {}^c\delta(l) = \delta({}^a l) \\ {}^{cg}\delta(l) = {}^g\delta({}^b l) \end{cases} \right\} \\ &\leq \{(a,b)) \in L\langle u\rangle \times L\langle u\rangle \mid \forall l \in L,\ {}^a\big({i_g}^\delta(l)\big) = {i_g}^\delta({}^b l)\}. \end{aligned}$$

The last group is the normalizer of $\Delta(L, {i_g}^\delta, L)$ in $L\langle u\rangle \times L\langle u\rangle$. Conversely, if $(a,b) \in N_{L\langle u\rangle \times L\langle u\rangle}(\Delta(L, {i_g}^\delta, L))$, since $(Q,\delta) \in \mathcal{P}(G,L,u)$, there exists $c \in N_G(Q)$ such that $(c,a) \in N_{Q,\delta}$. In other words, we have

$$\forall l \in L,\ {}^a\big({i_g}^\delta(l)\big) = {i_g}^\delta({}^b l) \text{ and } {}^c\delta(l) = \delta({}^a l).$$

It follows that

$$\forall l \in L,\ i_g\delta({}^b l) = \delta\big({}^a {i_g}^\delta(l)\big) = {}^c\delta {i_g}^\delta(l) = {}^c i_g\delta(l),$$

that is $(c,b) \in N_{Q,i_g\delta}$. Thus $(a,b) \in N^o_{Q,\delta} * N_{Q,i_g\delta}$, and this gives

$$N^o_{Q,\delta} * N_{Q,i_g\delta} = N_{L\langle u\rangle \times L\langle u\rangle}\big(\Delta(L, {i_g}^\delta, L)\big).$$

To simplify the notation, we denote this group by $N_{L,u}({i_g}^\delta)$.

We also observe that $\Delta(L, {i_g}^\delta, L) = \Delta^o(Q,\delta,L) * \Delta(Q, i_g\delta, L)$. We denote this group by $\Delta_{L,u}({i_g}^\delta)$, and we set

$$\overline{N}_{L,u}({i_g}^\delta) = N_{L,u}({i_g}^\delta)/\Delta_{L,u}({i_g}^\delta).$$

Since (L,u) is a D^Δ-pair, this group fits into a short exact sequence of groups

$$1 \to Z(L) \to \overline{N}_{L,u}({i_g}^\delta) \to \langle u\rangle \to 1. \tag{5.15.1}$$

This sequence splits by the map sending $v \in \langle u\rangle$ to $\big(\hat\theta_g(v), v\big) \in \overline{N}_{L,u}({i_g}^\delta)$.

5.16. From the above discussion follows that

$$\pi(M) = \sum_{g \in \overline{G}_{Q,\delta}} \pi\left(\mathrm{Ind}_{N_{L,u}({i_g}^\delta)}^{L\langle u\rangle \times L\langle u\rangle}\big(\mathrm{Inf}_{\overline{N}^o_{Q,\delta}}^{N^o_{Q,\delta}} F^o \underset{kC_G(Q)}{\otimes} \mathrm{Inf}_{\overline{N}_{Q,i_g\delta}}^{N_{Q,i_g\delta}\,(g,1)} E\big)\right).$$

The subgroup $\Delta_{L,u}({i_g}^\delta)$ of $N_{L,u}({i_g}^\delta)$ acts trivially on the module

$$T(F,g,E) = \mathrm{Inf}_{\overline{N}^o_{Q,\delta}}^{N^o_{Q,\delta}} F^o \underset{kC_G(Q)}{\otimes} \mathrm{Inf}_{\overline{N}_{Q,i_g\delta}}^{N_{Q,i_g\delta}\,(g,1)} E,$$

so $T(F,g,E)$ is inflated from a $k\overline{N}_{L,u}({i_g}^\delta)$-module

$$T(F,g,E) = \mathrm{Inf}_{\overline{N}_{L,u}({i_g}^\delta)}^{N_{L,u}({i_g}^\delta)} \overline{T}(F,g,E),$$

where $\overline{T}(F,g,E) = F^o \underset{kC_G(Q)}{\otimes} {}^{(g,1)}E$.

The discussion in Section 5.13 above shows that the vertices of the indecomposable summands of $T(F,g,E)$ are equal to $\Delta_{L,u}(i_g{}^\delta)$. In other words $\overline{T}(F,g,E)$ is a projective $\overline{N}_{L,u}(i_g{}^\delta)$-module. In view of the split exact sequence (5.15.1), we have that $\overline{N}_{L,u}(i_g{}^\delta) \cong Z(L) \rtimes \langle u\rangle$, and

$$\overline{T}(F,g,E) \cong \bigoplus_{\lambda\in\langle u\rangle^\natural} m_\lambda(F,g,E)\mathrm{Ind}_{\langle u\rangle}^{Z(L)\rtimes\langle u\rangle} k_\lambda,$$

for some multiplicities $m_\lambda(F,g,E) \in \mathbb{N}$, where $\langle u\rangle^\natural$ is the set of group homomorphisms $\lambda : \langle u\rangle \to k^\times$. It is easy to check that

$$\begin{aligned} m_\lambda(F,g,E) &= \dim_k \mathrm{Hom}_{Z(L)\rtimes\langle u\rangle}\big(\mathrm{Inf}_{\langle u\rangle}^{Z(L)\rtimes\langle u\rangle} k_\lambda, \overline{T}(F,g,E)\big) \\ &= \dim_k \mathrm{Hom}_{\langle u\rangle}\Big(k_\lambda, \big(\overline{T}(F,g,E)\big)^{Z(L)}\Big). \end{aligned} \tag{5.16.1}$$

Now the image of the module $\mathrm{Ind}_{N_{L,u}(i_g{}^\delta)}^{L\langle u\rangle\times L\langle u\rangle}\mathrm{Inf}_{\overline{N}_{L,u}(i_g{}^\delta)}^{N_{L,u}(i_g{}^\delta)}\mathrm{Ind}_{\langle u\rangle}^{Z(L)\rtimes\langle u\rangle} k_\lambda$ in the essential algebra $\mathcal{E}_\mathbb{F}(L\langle u\rangle)$ is equal to $kG_{\hat{\theta}_g,\lambda^{-1}}$, where $\theta_g \in \mathrm{Out}(L\langle u\rangle)$ and $\hat{\theta}_g \in \mathrm{Aut}(L\langle u\rangle)$ are defined in Section 5.14[7]. So our relations $\mathcal{R}_{Q,\delta}$ of (5.10.3) become

$$\forall F \in \mathrm{Pim}(k\overline{N}_{Q,\delta}), \qquad \sum_{\substack{g\in\overline{G}_{Q,\delta}\\ \lambda\in\langle u\rangle^\natural \\ E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})}} m_\lambda(F,g,E)\, kG_{\theta_g,\lambda}\cdot v_E = 0. \tag{5.16.2}$$

5.17. In order to understand these relations, we are going to change them a little bit, by replacing F by its dual and the tensor product $-\otimes_{kC_G(Q)} -$ appearing in $T(F,g,E)$ by $\mathrm{Hom}_{kC_G(Q)}(-,-)$, in other words, by setting now

$$\overline{T}(F,g,E) = \mathrm{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E). \tag{5.17.1}$$

In particular, the action of $\overline{N}_{L,u}(i_g{}^\delta)$ on $\overline{T}(F,g,E)$ is given as follows: for $(a,b) \in N_{L,u}(i_g{}^\delta)$, choose $s \in G$ such that $(s,a) \in N_{Q,\delta}$. Then $(s^g,b) \in N_{Q,\delta}$, and for $\varphi \in \mathrm{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E)$, we have

$$\forall f \in F,\ \big((a,b)\varphi\big)(f) = (s^g,b)\varphi\big((s,a)^{-1}f\big). \tag{5.17.2}$$

The action of $Z(L)$ on $\overline{T}(F,g,E)$ is simply given by multiplication

$$\forall z \in Z(L), \forall\varphi \in \mathrm{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E), \forall f \in F,\ (z\varphi)(f) = \delta(z)^g\cdot\varphi(f),$$

[7]Note that for $g \in \widehat{G}_{Q,\delta}$, the outer automorphism θ_g only depends on $gG_{Q,\delta} \in \overline{G}_{Q,\delta}$.

where $\delta(z) \in Z(Q)$. It follows that

$$\begin{aligned}\overline{T}(F,g,E)^{Z(L)} &= \mathrm{Hom}_{kC_G(Q)}\big(F, {}^{(g,1)}E^{Z(Q)}\big)\\ &\cong \mathrm{Hom}_{kC_G(Q)/Z(Q)}\big(F^{Z(Q)}, {}^{(g,1)}E^{Z(Q)}\big),\end{aligned}$$

where the last isomorphism comes from the fact that since F is projective, the module of co-invariants $F_{Z(Q)}$ is isomorphic to the module of invariants $F^{Z(Q)}$.

Now in view of (5.16.1), we have to describe the action of u, that is the element $\big(\hat{\theta}_g(u), u\big)$ of $N_{L,u}(i_g{}^{\delta})$, on $\overline{T}(F,g,E)$. For this, we use (5.17.2), and we choose $s \in G$ such that $\big(s, \hat{\theta}_g(u)\big) \in N_{Q,\delta}$. Now for $\varphi \in \mathrm{Hom}_{C_G(Q)}(F, {}^{(g,1)}E)$, we have

$$\forall f \in F,\ (u\varphi)(f) = (s^g, u)\varphi\big((s, \hat{\theta}_g(u))^{-1} f\big).$$

For each $\lambda \in \langle u\rangle^\natural$, an element of $\mathrm{Hom}_{\langle u\rangle}\Big(k_\lambda, \mathrm{Hom}_{kC_G(Q)}\big(F^{Z(Q)}, {}^{(g,1)}E^{Z(Q)}\big)\Big)$ is now determined by an element $\varphi \in \mathrm{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E)$ such that

$$\forall f \in F,\ \lambda(u)\varphi(f) = (s^g, u)\varphi\Big(\big(s, \hat{\theta}_g(u)\big)^{-1} f\Big).$$

In other words

$$\forall f \in F,\ \varphi\Big(\big(s, \hat{\theta}_g(u)\big) f\Big) = \lambda(u)^{-1}(s^g, u)\varphi(f).$$

Since $(s^g, u) = \Phi_g^{-1}\big((s, \hat{\theta}_g(u)\big)$, we get that

$$\forall f \in F,\ \varphi\Big(\big(s, \hat{\theta}_g(u)\big) f\Big) = \lambda(u)^{-1}\Phi_g^{-1}\big((s, \hat{\theta}_g(u)\big)\varphi(f).$$

Now let $c \in C_G(Q)$. Then $(c,1) \in N_{Q,\delta}$, and

$$\forall f \in F,\ \varphi\big((c,1)f\big) = (c^g, 1)\varphi(f) = \lambda(1)^{-1}\Phi_g^{-1}\big((c,1)\big)\varphi(f).$$

Let $\overline{(a,b)}$ denote the image in $\overline{N}_{Q,\delta}$ of $(a,b) \in N_{Q,\delta}$. Now the element $\overline{\big(s, \hat{\theta}_g(u)\big)}$, together with the elements $\overline{(c,1)}$, for $c \in C_G(Q)$, generate the whole of $\overline{N}_{Q,\delta}$. It follows that for any $\overline{(a,b)} \in \overline{N}_{Q,\delta}$

$$\varphi\big(\overline{(a,b)}f\big) = \lambda\big(\hat{\theta}_g^{-1}(b)\big)^{-1}\bar{\Phi}_g^{-1}\big(\overline{(a,b)}\big)\varphi(f).$$

In other words φ is a morphism of $k\overline{N}_{Q,\delta}$-modules from F to the module ${}^{[g,\lambda]}E$, equal to E as a k-vector space, but with action of $\overline{(a,b)} \in \overline{N}_{Q,\delta}$ given by

$$\forall e \in E,\ \overline{(a,b)} \cdot e \text{ (in } {}^{[g,\lambda]}E) := \lambda\big(\hat{\theta}_g^{-1}(b)\big)^{-1}\bar{\Phi}_g^{-1}\big(\overline{(a,b)}\big) \cdot e \text{ (in } E). \qquad (5.17.3)$$

It follows that

$$m_\lambda(F,g,E) = \dim_k \mathrm{Hom}_{\overline{N}_{Q,\delta}^\flat}\big(F^{Z(Q)}, {}^{[g,\lambda]}E^{Z(Q)}\big),$$

so our relations (5.16.2) become

$$\forall F \in \mathrm{Pim}(k\overline{N}_{Q,\delta}), \quad \sum_{\substack{g\in\overline{G}_{Q,\delta}\\ \lambda\in\langle u\rangle^\natural\\ E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})}} \dim_k \mathrm{Hom}_{\overline{N}^\flat_{Q,\delta}}\big(F^{Z(Q)}, {}^{[g,\lambda]}E^{Z(Q)}\big) kG_{\theta_g,\lambda}\cdot v_E = 0. \tag{5.17.4}$$

Remark 5.18: It should be noted that in (5.17.3), the coefficient $\lambda\big(\hat{\theta}_g^{-1}(b)\big)^{-1}$ does not depend on the choice of $\hat{\theta}_g \in \mathrm{Aut}(L\langle u\rangle)$ in the class $\theta_g \in \mathrm{Out}(L\langle u\rangle)$, as two different choices differ by an inner automorphism, so the corresponding values $\hat{\theta}_g(b)$ are conjugate. Hence we could write $\lambda\big(\theta_g^{-1}(b)\big)^{-1}$ instead of $\lambda\big(\hat{\theta}_g^{-1}(b)\big)^{-1}$.

5.19. Suppose now that $g, h \in \widehat{G}_{Q,\delta}$, and $\lambda, \mu \in \langle u\rangle^\natural$. If E is a projective $k\overline{N}_{Q,\delta}$-module, we claim that the $k\overline{N}_{Q,\delta}$-modules ${}^{[h,\mu]}\big({}^{[g,\lambda]}E\big)$ and ${}^{[hg,(\mu\circ\hat{\theta}_g)\cdot\lambda]}E$ are isomorphic. Indeed, there exists an element $w \in Z(L)$ such that $\hat{\theta}_{hg} = \hat{\theta}_h\hat{\theta}_g i_w$. Now $(1,w) \in N_{Q,\delta}$, and we can define $f: E \to E$ by $f(e) = \overline{(1,w^{-1})}\cdot e$. Then f is clearly an automorphism of the k-vector space E.

Claim: *The map f is an isomorphism of $k\overline{N}_{Q,\delta}$-modules from ${}^{[h,\mu]}\big({}^{[g,\lambda]}E\big)$ to ${}^{[hg,(\mu\circ\hat{\theta}_g)\cdot\lambda]}E$.*

Proof: Indeed, for $e \in E$, let ${}^{[g,\lambda]}e$ denote the element e of ${}^{[g,\lambda]}E$. Then for $(a,b) \in N_{Q,\delta}$

$$\begin{aligned}\overline{(a,b)}\cdot {}^{[h,\mu]}\big({}^{[g,\lambda]}e\big) &= \mu\big(\hat{\theta}_h^{-1}(b)\big)\bar{\Phi}_h^{-1}\big(\overline{(a,b)}\big)\cdot {}^{[g,\lambda]}e\\ &= \mu\big(\hat{\theta}_h^{-1}(b)\big)\overline{\big(a^h,\hat{\theta}_h^{-1}(b)\big)}\cdot {}^{[g,\lambda]}e\\ &= \mu\big(\hat{\theta}_h^{-1}(b)\big)\lambda\big(\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b)\big)\overline{\big(a^{hg},\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b)\big)}\cdot e.\end{aligned}$$

It follows that

$$f\Big(\overline{(a,b)}\cdot {}^{[h,\mu]}\big({}^{[g,\lambda]}e\big)\Big) = \mu\big(\hat{\theta}_h^{-1}(b)\big)\lambda\big(\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b)\big)\overline{\big(a^{hg},w^{-1}\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b)\big)}\cdot e. \tag{5.19.1}$$

On the other hand

$$\begin{aligned}\overline{(a,b)}\cdot {}^{[hg,(\mu\circ\hat{\theta}_g)\cdot\lambda]}f(e) &= \big((\mu\circ\hat{\theta}_g)\cdot\lambda\big)\big(\hat{\theta}_{hg}^{-1}(b)\big)\overline{\big(a^{hg},\hat{\theta}_{hg}^{-1}(b)\big)}\cdot f(e)\\ &= \mu\big(\hat{\theta}_g\hat{\theta}_{hg}^{-1}(b)\big)\lambda\big(\hat{\theta}_{hg}^{-1}(b)\big)\overline{\big(a^{hg},\hat{\theta}_{hg}^{-1}(b)w^{-1}\big)}\cdot e\end{aligned} \tag{5.19.2}$$

Since $\hat{\theta}_{hg} = \hat{\theta}_h\hat{\theta}_g i_w$, we have $i_w^{-1}\hat{\theta}_g^{-1}\hat{\theta}_h^{-1} = \hat{\theta}_{hg}^{-1}$, so $\hat{\theta}_{hg}^{-1}(b)w^{-1} = w^{-1}\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b)$. Moreover

$$\lambda\big(\hat{\theta}_{hg}^{-1}(b)\big) = \lambda\big(w^{-1}\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b)w\big) = \lambda\big(\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b)\big),$$

and

$$\mu\big(\hat\theta_g\hat\theta_{hg}^{-1}(b)\big) = \mu\Big(\hat\theta_g\big(w^{-1}\hat\theta_g^{-1}\hat\theta_h^{-1}(b)w\big)\Big)$$
$$= \mu\Big(\hat\theta_g\big(w^{-1}\big)\hat\theta_h^{-1}(b)\theta_g(w)\Big) = \mu\big(\hat\theta_h^{-1}(b)\big).$$

Now it follows from (5.19.1) and (5.19.2) that

$$f\Big(\overline{(a,b)}\cdot {}^{[h,\mu]}\big({}^{[g,\lambda]}e\big)\Big) = \overline{(a,b)}\cdot {}^{[hg,(\mu\circ\hat\theta_g)\cdot\lambda]}f(e),$$

proving the claim. □

In addition to the above claim, we observe that if $g\in\widehat{G}_{Q,\delta}$ is actually in $G_{Q,\delta}$, then the $k\overline{N}_{Q,\delta}$-modules E and ${}^{[g,1]}E$ are isomorphic: Indeed, saying that $g\in G_{Q,\delta}$ amounts to saying that $\hat\theta_g$ is an inner automorphism i_x of $L\langle u\rangle$, for some $x\in L\langle u\rangle$ with $(g,x)\in N_{Q,\delta}$. Then (5.17.3) becomes

$$\forall e\in E,\ \overline{(a,b)}\cdot e\ (\text{in } {}^{[g,1]}E) = \bar\Phi_g^{-1}\big(\overline{(a,b)}\big)\cdot e\ (\text{in } E)$$
$$= \overline{(a^g,b^x)}\cdot e.$$

Then the map $e\in E\mapsto \overline{(g^{-1},x^{-1})}\cdot e\in {}^{[g,1]}E$ is an isomorphism of $k\overline{N}_{Q,\delta}$-modules.

We can now introduce the semidirect product

$$\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural := \big(\widehat{G}_{Q,\delta}/G_{Q,\delta}\big)\ltimes\langle u\rangle^\natural.$$

As a set $\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural$ is the cartesian product $\overline{G}_{Q,\delta}\times\langle u\rangle^\natural$. For $g\in G_{Q,\delta}$ and $\lambda\in\langle u\rangle^\natural$, let $[g,\lambda]$ denote the pair $(gG_{Q,\delta},\lambda)$ in $\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural$. The product in $\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural$ is given as follows: For $g,h\in G_{Q,\delta}$ and $\lambda,\mu\in\langle u\rangle^\natural$, we have

$$[h,\mu][g,\lambda] = [hg,(\mu\circ\theta_g)\cdot\lambda].$$

The above discussion now shows that there is an action of $\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural$ on the group $\mathrm{Proj}(k\overline{N}_{Q,\delta})$. The group $\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural$ also acts on the set $\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})$ introduced at Section 5.9: Indeed the restriction of ${}^{[g,\lambda]}E$ to $C_G(Q)$ is isomorphic to the restriction of ${}^{[g,1]}E$, and $\widehat{G}_{Q,\delta}$ permutes the indecomposable $kC_G(Q)$-modules.

5.20. Yet another (well known) lemma[8]:

Lemma 5.21: *Let H be a finite group, and R be a normal p-subgroup of H.*

1. *The assignment $E\mapsto E^R$ induces a bijection between the set of isomorphism classes of indecomposable projective kH-modules and the set of isomorphism classes of indecomposable projective $k(H/R)$-modules.*

[8]We will not use Assertion 2 of this lemma here, but we state it for completeness.

2. Moreover if R is central in H, then for any projective kH-modules E and F

$$\dim_k \mathrm{Hom}_{kH}(E,F) = |R| \dim_{k(H/R)}(E^R, F^R).$$

Proof: 1. Let E be a projective kH-module, and M be any $k(H/R)$-module. Then

$$\mathrm{Hom}_{kH}(E, \mathrm{Inf}_{H/R}^H M) \cong \mathrm{Hom}_{k(H/R)}(E_R, M) \cong \mathrm{Hom}_{k(H/R)}(E^R, M)$$

as $E_R \cong E^R$ if E is projective. Now the functor $M \mapsto \mathrm{Hom}_{kH}(E, \mathrm{Inf}_{H/R}^H M)$ is exact, since inflation is exact, so E^R is a projective $k(H/R)$-module. Moreover, the simple kH-modules are inflated from H/R, and the previous isomorphism shows that if E is indecomposable, then E^R has a unique simple $k(H/R)$-quotient, thus it is indecomposable.

2. If now R is central in H, then R acts on $\mathrm{Hom}_{kH}(E,F)$ by left multiplication, that is $(r\varphi)(e) = r \cdot \varphi(e)$ for $r \in R$, $e \in E$, and $\varphi \in \mathrm{Hom}_{kH}(E,F)$. Moreover, this action is free, since if $E = F = kH$, then $\mathrm{Hom}_{kH}(E,F) \cong kH$ is free as a kR-module. Thus

$$\begin{aligned}\dim_k \mathrm{Hom}_{kH}(E,F) &= |R| \dim_k \big(\mathrm{Hom}_{kH}(E,F)\big)^R = |R| \dim_k \mathrm{Hom}_{kH}(E, F^R)\\ &= |R| \dim_k \mathrm{Hom}_{k(H/R)}(E_R, F^R)\\ &= |R| \dim_k \mathrm{Hom}_{k(H/R)}(E^R, F^R),\end{aligned}$$

as was to be shown. □

5.22. Let $\mathbb{F}R_k(\overline{N}_{Q,\delta}^\flat) = \mathbb{F} \otimes_{\mathbb{Z}} R_k(\overline{N}_{Q,\delta}^\flat)$ denote the Grothendieck group of finite dimensional $k\overline{N}_{Q,\delta}^\flat$-modules, extended by $\mathbb{F}$. For a projective $k\overline{N}_{Q,\delta}^\flat$-module X, let $[X]$ denote its image in $\mathbb{F}R_k(\overline{N}_{Q,\delta}^\flat)$. The group $\overline{G}_{Q,\delta} \ltimes \langle u \rangle^\natural$ also acts on $R_k(\overline{N}_{Q,\delta}^\flat)$, by permutation of its base consisting of the isomorphism classes of simple modules.

We denote by $\Gamma_{Q,\delta}$ the image of the linear map

$$\gamma_{Q,\delta}^{L,u} : \mathbb{F}\mathrm{Proj}^\sharp(k\overline{N}_{Q,\delta}) \to \mathbb{F}R_k(\overline{N}_{Q,\delta}^\flat)$$

induced by $E \mapsto [E^{Z(Q)}]$. The map $\gamma_{Q,\delta}^{L,u}$ is a map of $\mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^\natural)$-modules, so its image $\Gamma_{Q,\delta}$ is also a $\mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^\natural)$-module.

The $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$-module V is also a $\mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^\natural)$-module, thanks to the (surjective) homomorphism of algebras sending $[g,\lambda] \in \mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^\natural)$ to the image of $kG_{\theta_g,\lambda}$ in $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$. Tensoring with V gives a surjective map

$$\gamma_{Q,\delta}^{L,u} \otimes \mathrm{Id}_V : \mathbb{F}\mathrm{Proj}^\sharp(k\overline{N}_{Q,\delta}) \otimes_{\mathbb{F}} V \to \Gamma_{Q,\delta} \otimes_{\mathbb{F}} V,$$

of $\mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u\rangle^\natural)$-modules, where the action of $\overline{G}_{Q,\delta} \ltimes \langle u\rangle^\natural$ on both tensor products is diagonal.

Now rephrasing (5.10.3), we get a surjective map

$$\sigma = \bigoplus \sigma_{Q,\delta} : \bigoplus_{(Q,\delta)\in[\mathcal{P}(G,l,u)]} \mathbb{F}\mathrm{Proj}^\sharp(k\overline{N}_{Q,\delta}) \otimes_{\mathbb{F}} V \to S_{L\langle u\rangle,V}(G),$$

where $\sigma_{Q,\delta}$ sends $E\otimes v \in \mathbb{F}\mathrm{Proj}^\sharp(k\overline{N}_{Q,\delta}) \otimes_k V$ to the image of $T(Q,\delta,E)\otimes v$ in $S_{L\langle u\rangle,V}(G)$.

The kernel of this map is the direct sum for $(Q,\delta) \in [\mathcal{P}(G,L,u)]$ of the kernels of its components $\sigma_{Q,\delta}$. By (5.17.4), the sum $\sum\limits_{E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})} E\otimes v_E$ is in the kernel of $\sigma_{Q,\delta}$ if and only if

$$\forall F \in \mathrm{Pim}(k\overline{N}_{Q,\delta}),\ \sum_{\substack{[g,\lambda]\in\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural\\ E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})}} \dim_k \mathrm{Hom}_{\overline{N}^\flat_{Q,\delta}}\big(F^{Z(Q)}, {}^{[g,\lambda]}E^{Z(Q)}\big)[g,\lambda]\cdot v_E = 0. \tag{5.22.1}$$

Now $R_k(\overline{N}^\flat_{Q,\delta})$ has a basis consisting of the $[X_F]$ for $F\in\mathrm{Pim}(k\overline{N}_{Q,\delta})$, where X_F is the unique simple quotient of the $k\overline{N}^\flat_{Q,\delta}$-module $F^{Z(Q)}$. So (5.22.1) is equivalent to

$$\sum_{F\in\mathrm{Pim}(k\overline{N}_{Q,\delta})} [X_F]\otimes \sum_{\substack{[g,\lambda]\in\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural\\ E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})}} \dim_k \mathrm{Hom}_{\overline{N}^\flat_{Q,\delta}}\big(F^{Z(Q)}, {}^{[g,\lambda]}E^{Z(Q)}\big)[g,\lambda]\cdot v_E = 0$$

in $\mathbb{F}R_k(\overline{N}_{Q,\delta})\otimes V$. This in turn is equivalent to

$$\sum_{F\in\mathrm{Pim}(k\overline{N}_{Q,\delta})} \sum_{\substack{[g,\lambda]\in\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural\\ E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})}} \dim_k \mathrm{Hom}_{\overline{N}^\flat_{Q,\delta}}\big(F^{Z(Q)}, {}^{[g,\lambda]}E^{Z(Q)}\big)[X_F]\otimes[g,\lambda]\cdot v_E = 0.$$

Now for any $[g,\lambda]\in\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural$ and any $E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})$

$$\sum_{F\in\mathrm{Pim}(k\overline{N}_{Q,\delta})} \dim_k \mathrm{Hom}_{\overline{N}^\flat_{Q,\delta}}\big(F^{Z(Q)}, {}^{[g,\lambda]}E^{Z(Q)}\big)[X_F] = [{}^{[g,\lambda]}E^{Z(Q)}] = {}^{[g,\lambda]}\gamma^{L,u}_{Q,\delta}(E).$$

It follows that $\sum\limits_{E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})} E\otimes v_E$ is in the kernel of $\sigma_{Q,\delta}$ if and only if

$$\sum_{E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})} \sum_{[g,\lambda]\in\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural} {}^{[g,\lambda]}\gamma^{L,u}_{Q,\delta}(E)\otimes[g,\lambda]\cdot v_E = 0$$

in $\mathbb{F}R_k(\overline{N}_{Q,\delta})\otimes V$. In other words $\sigma_{Q,\delta}$ has the same kernel as the map

$$\sum_{E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})} E\otimes v_E \mapsto \sum_{[g,\lambda]\in\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural} [g,\lambda]\big(\gamma^{L,u}_{Q,\delta}\otimes\mathrm{Id}\big)\Big(\sum_{E\in\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})} E\otimes v_E\Big).$$

It follows that the image of $\sigma_{Q,\delta}$ is isomorphic to the image of the previous map, that is

$$\mathrm{Im}(\sigma_{Q,\delta}) \cong \mathrm{Tr}_1^{\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural}\big(\Gamma_{Q,\delta}\otimes_{\mathbb{F}} V\big).$$

We finally get:

Theorem 5.23: *There is an isomorphism of k-vector spaces*

$$S_{L\langle u\rangle,V}(G) \cong \bigoplus_{(Q,\delta)\in[\mathcal{P}(G,L,u)]} \mathrm{Tr}_1^{\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural}\big(\Gamma_{Q,\delta}\otimes_{\mathbb{F}} V\big).$$

5.24. The simple diagonal p-permutation functors. Now we use Theorem 5.23 to describe the simple functors. We assume that $\mathbb{F}$ is algebraically closed, of characteristic 0 or p. Recall that we have an isomorphism of algebras

$$\mathcal{E}_{\mathbb{F}}\big(L\langle u\rangle\big) \cong \mathrm{Out}\big(L\langle u\rangle\big) \ltimes \mathbb{F}\overline{R}_k(\langle u\rangle).$$

In order to describe the simple $\mathcal{E}_{\mathbb{F}}(L\langle u\rangle)$-modules, we set $A = \mathbb{F}\overline{R}_k(\langle u\rangle)$ and $\Omega = \mathrm{Out}(L\langle u\rangle)$, and we use the results of [12], in our specific situation, as in Section 4.2 of [10]. First, the simple A-modules are one dimensional, of the form $\mathbb{F}_x$, where x is a generator of $\langle u\rangle$: For such a generator x, we get an algebra homomorphism $e_x : A = \mathbb{F}\overline{R}_k(\langle u\rangle) \to \mathbb{F}$ defined by

$$\forall \lambda \in \langle u\rangle^\natural,\ e_x(\lambda) = \widetilde{\lambda}(x),$$

where $\widetilde{\lambda} : \langle u\rangle \to \mathbb{F}^\times$ lifts λ. This makes sense because if λ is induced from a proper subgroup of $\langle u\rangle$, then $\widetilde{\lambda}(x) = 0$, as x cannot be contained in a proper subgroup of $\langle u\rangle$. So e_x extends to an algebra homomorphism $A = \mathbb{F}\overline{R}_k(\langle u\rangle) \to \mathbb{F}$, which in turn yields a one dimensional A-module $\mathbb{F}_x$. We get all the simple A-modules in this way.

We also abusively denote by $\widetilde{\lambda}$ the composition $L\langle u\rangle \longrightarrow \langle u\rangle \xrightarrow{\widetilde{\lambda}} \mathbb{F}^\times$.

Now the stabilizer Ω_x of $\mathbb{F}_x$ in $\Omega = \mathrm{Out}(L\langle u\rangle)$ is the set of classes of $\gamma \in \mathrm{Aut}(L\langle u\rangle)$ such that $\gamma(x) = x$. This does not depend on the generator x of $\langle u\rangle$, and it is equal to $\mathrm{Out}(L,u)$. We note that Ω_x acts trivially on A, since it acts trivially on $\langle u\rangle$, so $\Omega_x \ltimes A = \Omega_x \times A$.

So any simple $\mathcal{E}_{\mathbb{F}}\big(L\langle u\rangle\big)$-module V is of the form

$$V = \mathrm{Ind}_{\Omega_x\times A}^{\Omega\ltimes A}(W\otimes \mathbb{F}_x),$$

for some generator x of $\langle u\rangle$ and some simple $\mathbb{F}\Omega_x$-module W. The action of $\Omega_x \times A$ on $W\otimes\mathbb{F}_x$ is given by

$$\forall \gamma\in\Omega_x,\ \forall a\in A,\ \forall w\in W,\ (\gamma,a)\cdot(w\otimes 1) = e_x(a)\big((\gamma\cdot w)\otimes 1\big).$$

Moreover, the isomorphism type of the simple $\mathcal{E}_{\mathbb{F}}(L\langle u\rangle)$-module V is determined by the isomorphism type of the simple $\mathbb{F}\mathrm{Out}(L,u)$-module W, and the choice of the generator x of $\langle u\rangle$, up to the action of Ω, i.e. up to the action of the subgroup $\mathrm{Aut}(L\langle u\rangle)^\sharp$ of $\mathrm{Aut}(L\langle u\rangle)$ consisting of automorphisms which stabilize $\langle u\rangle$.

Then (L,x) is a D^Δ-pair, such that $\mathrm{Out}(L,u)=\mathrm{Out}(L,x)$, and choosing x up to the action of $\mathrm{Aut}(L\langle u\rangle)^\sharp$ amounts to choosing (L,x) in a set of D^Δ-pairs such that $L\langle x\rangle = L\langle u\rangle$, up to isomorphism of D^Δ-pairs. So up to changing (L,u) to (L,x), we can parametrize the simple functor $S_{L\langle u\rangle,V}$ by the triple (L,x,W) instead, that is, we can suppose $x=u$ in the previous calculations, and set $\mathsf{S}_{L,u,W}=S_{L\langle u\rangle,V}$, where

$$V=\mathrm{Ind}_{\mathrm{Out}(L,u)\times\mathbb{F}\overline{R}_k(\langle u\rangle)}^{\mathrm{Out}(L\langle u\rangle)\ltimes\mathbb{F}\overline{R}_k(\langle u\rangle)}(W\otimes\mathbb{F}_u).$$

By Theorem 5.23, we have that

$$\mathsf{S}_{L,u,W}(G)\cong\bigoplus_{(Q,\delta)\in[\mathcal{P}(G,L,u)]}\mathrm{Tr}_{\mathbf{1}}^{\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural}(\Gamma_{Q,\delta}\otimes_{\mathbb{F}}V),$$

so we must describe the action of $\overline{G}_{Q,\delta}\ltimes\langle u\rangle^\natural$ on V. We recall from 5.14 that the group homomorphism

$$g\in\widehat{G}_{Q,\delta}\to\theta_g\in\mathrm{Out}(L\langle u\rangle)$$

induces an embedding $\overline{G}_{Q,\delta}\hookrightarrow\mathrm{Out}(L\langle u\rangle)$, and we identify $\overline{G}_{Q,\delta}$ with its image via this embedding.

Now the semidirect product $\mathrm{Out}(L\langle u\rangle)\ltimes\langle u\rangle^\natural$ embeds in $\mathrm{Out}(L\langle u\rangle)\ltimes\mathbb{F}\overline{R}_k(\langle u\rangle)$ via $[\theta,\lambda]\mapsto\theta\ltimes\overline{\lambda}$, for $\theta\in\mathrm{Out}(L\langle u\rangle)$ and $\lambda\in\langle u\rangle^\natural$. In particular, it acts on V. We observe moreover that there is an isomorphism of $\mathbb{F}$-vector spaces

$$V\cong\bigoplus_{\psi\in\mathrm{Out}(L\langle u\rangle)/\mathrm{Out}(L,u)}(\psi\otimes W). \tag{*}$$

With this decomposition, for $\theta\in\mathrm{Aut}(L\langle u\rangle)$, $\lambda\in\langle u\rangle^\natural$, $\psi\in\mathrm{Out}(L\langle u\rangle)/\mathrm{Out}(L,u)$, and $w\in W$, we have

$$\begin{aligned}
[\theta,\lambda]\cdot(\psi\otimes w)&=(\theta\ltimes\overline{\lambda})(\psi\ltimes\overline{1})\cdot(\mathrm{Id}\otimes w)\\
&=\big((\theta\circ\psi)\ltimes(\overline{\lambda\circ\psi})\big)\cdot(\mathrm{Id}\otimes w)\\
&=\big((\theta\circ\psi)\ltimes\overline{1}\big)\big(\mathrm{Id}\ltimes(\overline{\lambda\circ\psi})\big)\cdot(\mathrm{Id}\otimes w)\\
&=\widetilde{\lambda}\big(\psi(u)\big)\big((\theta\circ\psi)\ltimes\overline{1}\big)\cdot(\mathrm{Id}\otimes w)\\
&=\widetilde{\lambda}\big(\psi(u)\big)(\theta\ltimes\overline{1})(\psi\ltimes\overline{1})(\mathrm{Id}\otimes w)\\
&=\widetilde{\lambda}\big(\psi(u)\big)\,(\theta\ltimes\overline{1})\cdot(\psi\otimes w)\\
&=\widetilde{\lambda}\big(\psi(u)\big)\,\theta\cdot(\psi\otimes w),
\end{aligned}$$

where $\psi \otimes w \mapsto \theta\cdot(\psi \otimes w)$ denotes the action of θ on $\psi \otimes w \in \mathrm{Ind}_{\mathrm{Out}(L,u)}^{\mathrm{Out}(L\langle u\rangle)} W$.

So the group $H := \mathrm{Out}(L\langle u\rangle) \ltimes \langle u\rangle^\natural$ permutes the components $\psi \otimes W$ of the direct sum $(*)$, and it permutes them transitively. The stabilizer of the component $\mathrm{Id}\otimes W$ is equal to $H_1 := \mathrm{Out}(L,u) \ltimes \langle u\rangle^\natural$, and $H_1 = \mathrm{Out}(L,u) \times \langle u\rangle^\natural$ as $\mathrm{Out}(L,u)$ acts trivially on $\langle u\rangle^\natural$. The group H_1 acts on $\mathrm{Id}\otimes W$ by

$$\forall(\theta,\lambda) \in H_1,\ (\theta,\lambda)\cdot(\mathrm{Id}\otimes w) = \widetilde{\lambda}(u)(\mathrm{Id}\otimes \theta\cdot w).$$

It follows that there is an isomorphism of $\mathbb{F}H$-modules

$$V \cong \mathrm{Ind}_{H_1}^{H} W,$$

where the action of $(\theta,\lambda) \in H_1$ on W is given by

$$(\theta,\lambda)\cdot w = \widetilde{\lambda}(u)\,\theta\cdot w.$$

Now we consider the restriction of V to $K_{Q,\delta} := \overline{G}_{Q,\delta} \ltimes \langle u\rangle^\natural$, and use the Mackey formula. The map

$$\psi \in \mathrm{Out}(L\langle u\rangle) \mapsto K_{Q,\delta}(\psi \ltimes 1)H_1$$

induces a bijection from $\overline{G}_{Q,\delta}\backslash\mathrm{Out}(L\langle u\rangle)/\mathrm{Out}(L,u)$ to $K_{Q,\delta}\backslash H/H_1$. This gives

$$\mathrm{Res}_{K_{Q,\delta}}^{H} V \cong \bigoplus_{\psi\in[\overline{G}_{Q,\delta}\backslash\mathrm{Out}(L\langle u\rangle)/\mathrm{Out}(L,u)]} \mathrm{Ind}_{O_\psi}^{K_{Q,\delta}}\ {}^{(\psi\ltimes 1)}\mathrm{Res}_{{}_\psi O}^{H_1} W, \qquad (*)$$

where

$$\begin{aligned}{}_\psi O &= {K_{Q,\delta}}^{\psi\ltimes 1} \cap H_1\\ O_\psi &= K_{Q,\delta} \cap {}^{\psi\ltimes 1}H_1 = {}^{\psi\ltimes 1}({}_\psi O).\end{aligned}$$

Now ${K_{Q,\delta}}^{\psi\ltimes 1} = (\overline{G}_{Q,\delta})^\psi \ltimes \langle u\rangle^\natural$. Moreover

$$\begin{aligned}(\overline{G}_{Q,\delta})^\psi &= \big\{\theta^\psi \mid \theta \in \mathrm{Out}(L\langle u\rangle),\ \exists g \in N_G(Q),\ \theta_{|L} = (i_g)^\delta\big\}\\ &= \big\{\theta \in \mathrm{Out}(L\langle u\rangle) \mid \exists g \in N_G(Q),\ ({}^\psi\theta)_{|L} = (i_g)^\delta\big\}\\ &= \big\{\theta \in \mathrm{Out}(L\langle u\rangle) \mid \exists g \in N_G(Q),\ \theta_{|L} = (i_g)^{\delta\psi}\big\}\\ &= \overline{G}_{Q,\delta\psi}.\end{aligned}$$

Hence

$${}_\psi O = \big(\overline{G}_{\delta\psi} \ltimes \langle u\rangle^\natural\big) \cap \big(\mathrm{Out}(L,u) \times \langle u\rangle^\natural\big) = \overline{G}_{Q,\delta\psi,u} \times \langle u\rangle^\natural,$$

where we have set

$$\overline{G}_{Q,\delta\psi,u} = \big\{\theta \in \mathrm{Out}(L,u) \mid \exists g \in N_G(Q),\ \theta_{|L} = (i_g)^{\delta\psi}\big\}.$$

From (*), and omitting the appropriate restriction symbols before $\Gamma_{Q,\delta}$, we now get

$$\begin{aligned}
\mathrm{Tr}_{\mathbf{1}}^{K_{Q,\delta}}\big(\Gamma_{Q,\delta}\otimes\mathrm{Res}^{H}_{K_{Q,\delta}}V\big) &\cong \bigoplus_{\psi\in[\overline{G}_{Q,\delta}\backslash\mathrm{Out}(L\langle u\rangle)/\mathrm{Out}(L,u)]} \mathrm{Tr}_{\mathbf{1}}^{\psi\ltimes 1({}_{\psi}O)}\ \big(\Gamma_{Q,\delta}\otimes{}^{(\psi\ltimes 1)}\mathrm{Res}^{H_1}_{{}_{\psi}O}W\big)\\
&\cong \bigoplus_{\psi\in[\overline{G}_{Q,\delta}\backslash\mathrm{Out}(L\langle u\rangle)/\mathrm{Out}(L,u)]} \mathrm{Tr}_{\mathbf{1}}^{{}_{\psi}O}\ \big((\Gamma_{Q,\delta})^{\psi\ltimes 1}\otimes\mathrm{Res}^{H_1}_{{}_{\psi}O}W\big)\\
&\cong \bigoplus_{\psi\in[\overline{G}_{Q,\delta}\backslash\mathrm{Out}(L\langle u\rangle)/\mathrm{Out}(L,u)]} \mathrm{Tr}_{\mathbf{1}}^{{}_{\psi}O}\ \big(\Gamma_{Q,\delta\psi}\otimes\mathrm{Res}^{H_1}_{{}_{\psi}O}W\big).
\end{aligned}$$

This finally gives

$$\begin{aligned}
\mathsf{S}_{L,u,W}(G) &\cong \bigoplus_{\substack{(Q,\delta)\in[G\backslash\mathcal{P}(G,L,u)/\mathrm{Out}(L\langle u\rangle)]\\ \psi\in[\overline{G}_{Q,\delta}\backslash\mathrm{Out}(L\langle u\rangle)/\mathrm{Out}(L,u)]}} \mathrm{Tr}_{\mathbf{1}}^{\overline{G}_{Q,\delta\psi,u}\times\langle u\rangle^{\natural}}\big(\Gamma_{Q,\delta\psi}\otimes\mathrm{Res}^{\mathrm{Out}(L,u)\times\langle u\rangle^{\natural}}_{\overline{G}_{Q,\delta\psi,u}\times\langle u\rangle^{\natural}}W\big)\\
&\cong \bigoplus_{(Q,\delta)\in[G\backslash\mathcal{P}(G,L,u)/\mathrm{Out}(L,u)]} \mathrm{Tr}_{\mathbf{1}}^{\overline{G}_{Q,\delta,u}\times\langle u\rangle^{\natural}}\big(\Gamma_{Q,\delta}\otimes\mathrm{Res}^{\mathrm{Out}(L,u)\times\langle u\rangle^{\natural}}_{\overline{G}_{Q,\delta,u}\times\langle u\rangle^{\natural}}W\big)\\
&= \bigoplus_{(Q,\delta)\in[G\backslash\mathcal{P}(G,L,u)/\mathrm{Out}(L,u)]} \mathrm{Tr}_{\mathbf{1}}^{\overline{G}_{Q,\delta,u}\times\langle u\rangle^{\natural}}\big(\Gamma_{Q,\delta}\otimes W\big),
\end{aligned}$$

where we have omitted the restriction symbol $\mathrm{Res}^{\mathrm{Out}(L,u)\times\langle u\rangle^{\natural}}_{\overline{G}_{Q,\delta,u}\times\langle u\rangle^{\natural}}$ in the last line.

Moreover $\mathrm{Tr}_{\mathbf{1}}^{\overline{G}_{Q,\delta,u}\times\langle u\rangle^{\natural}}=\mathrm{Tr}_{\mathbf{1}}^{\overline{G}_{Q,\delta}}\circ\mathrm{Tr}_{\mathbf{1}}^{\langle u\rangle^{\natural}}$, so we first compute $\mathrm{Tr}_{\mathbf{1}}^{\langle u\rangle^{\natural}}(\Gamma_{Q,\delta}\otimes W)$. Let $\gamma\in\Gamma_{Q,\delta}$ and $w\in W$. Then

$$\begin{aligned}
\mathrm{Tr}_{\mathbf{1}}^{\langle u\rangle^{\natural}}(\gamma\otimes w) &= \sum_{\lambda\in\langle u\rangle^{\natural}}[1,\lambda]\cdot(\gamma\otimes w)\\
&= \sum_{\lambda\in\langle u\rangle^{\natural}}([1,\lambda]\cdot\gamma)\otimes([1,\lambda]\cdot w)\\
&= \Big(\sum_{\lambda\in\langle u\rangle^{\natural}}\widetilde{\lambda}(u)[1,\lambda]\cdot\gamma\Big)\otimes w. \qquad (**)
\end{aligned}$$

Let

$$\Xi_{Q,\delta}:=\big\{m\in\Gamma_{Q,\delta}\mid \forall\lambda\in\langle u\rangle^{\natural},\ [1,\lambda]\cdot m=\widetilde{\lambda}(u)^{-1}m\big\}.$$

Then one checks easily that the map

$$\gamma\in\Gamma_{Q,\delta}\mapsto\frac{1}{|u|}\sum_{\lambda\in\langle u\rangle^{\natural}}\widetilde{\lambda}(u)[1,\lambda]\cdot\gamma$$

is an idempotent endomorphism of $\Gamma_{Q,\delta}$, with image $\Xi_{Q,\delta}$. It follows from (**) that

$$\mathrm{Tr}_{\mathbf{1}}^{\langle u\rangle^\natural}(\Gamma_{Q,\delta}\otimes W)=\Xi_{Q,\delta}\otimes W.$$

Hence

$$\mathsf{S}_{L,u,W}(G)\cong\bigoplus_{(Q,\delta)\in[G\backslash\mathcal{P}(G,L,u)/\mathrm{Out}(L,u)]}\mathrm{Tr}_{\mathbf{1}}^{\overline{G}_{Q,\delta}}(\Xi_{Q,\delta}\otimes_{\mathbb{F}}W).$$

Now the space $\Xi_{Q,\delta}$ is the image by the map $\gamma_{Q,\delta}^{L,u}$ of the corresponding subspace

$$\mathbb{F}\mathrm{Proj}^\sharp(k\overline{N}_{Q,\delta})^0=\{n\in\mathbb{F}\mathrm{Proj}^\sharp(k\overline{N}_{Q,\delta})\mid\forall\lambda\in\langle u\rangle^\natural,\ n_\lambda=\widetilde{\lambda}(u)^{-1}n\}$$

of $\mathbb{F}\mathrm{Proj}^\sharp(k\overline{N}_{Q,\delta})$. This space has a basis consisting of the sums $\sum\limits_{\lambda\in\langle u\rangle^\natural}\lambda(u)E_\lambda$, where E runs through a set Σ of representatives of orbits of $\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})$ under the action of $\langle u\rangle^\natural$. Since $\mathrm{Pim}^\sharp(k\overline{N}_{Q,\delta})$ is a $(\overline{G}_{Q,\delta,u},\langle u\rangle^\natural)$-biset, this set of representatives can moreover be chosen invariant by the action of $\overline{G}_{Q,\delta,u}$.

Now the restriction map $\mathrm{Res}_{C_G(Q)}^{\overline{N}_{Q,\delta}}$ induces a bijection from Σ to the set $\mathrm{Pim}\big(kC_G(Q),u\big)$ of isomorphism classes of u-invariant indecomposable projective $kC_G(Q)$-modules, and this bijection is $\overline{G}_{Q,\delta,u}$-equivariant. Moreover, taking fixed points by $Z(Q)$ gives a bijection from $\mathrm{Pim}\big(kC_G(Q),u\big)$ to the set $\mathrm{Pim}\big(kC_G(Q)/Z(Q),u\big)$ of isomorphism classes of u-invariant indecomposable projective $kC_G(Q)/Z(Q)$-modules.

Finally, we have proved the following:

Theorem 5.25: *Let $\mathbb{F}$ be a field of characteristic 0 or p.*

1. *The simple diagonal p-permutation functors over $\mathbb{F}$ are parametrized by triples (L,u,W), where (L,u) is a D^Δ-pair and W is a simple $\mathbb{F}\mathrm{Out}(L,u)$-module.*

2. *The evaluation at a finite group G of the simple functor $\mathsf{S}_{L,u,W}$ parametrized by the triple (L,u,W) is*

$$\mathsf{S}_{L,u,W}(G)\cong\bigoplus_{(Q,\delta)\in[G\backslash\mathcal{P}(G,L,u)/\mathrm{Out}(L,u)]}\mathrm{Tr}_1^{\overline{G}_{Q,\delta,u}}\Big(\mathbb{F}\mathrm{Cart}\big(kC_G(Q)/Z(Q),u\big)\otimes_{\mathbb{F}}W\Big),$$

where $\mathbb{F}\mathrm{Cart}\big(kC_G(Q)/Z(Q),u\big)$ is the image of the map

$$\mathbb{F}\mathrm{Pim}\big(kC_G(Q)/Z(Q),u\big)\to\mathbb{F}R_k\big(C_G(Q)/Z(Q)\big)$$

induced by the Cartan map.

6 Examples

6.1. The functor $\mathsf{S}_{\mathbf{1},1,\mathbb{F}}$. We apply Theorem 5.25 to the case where $L=\mathbf{1}$, so

$u = 1$, and $W = \mathbb{F}$. For a finite group G, we get that $(Q,\delta) \in \mathcal{P}(G,\mathbf{1},1)$ if and only if $Q = \mathbf{1}$ and $\delta : L \to Q$ is the identity. Moreover $G_{Q,\delta,1} = G = G_{Q,\delta}$, so by Theorem 5.25, we get that

$$\mathsf{S}_{\mathbf{1},1,\mathbb{F}}(G) \cong \mathbb{F}\mathrm{Cart}(G),$$

where $\mathbb{F}\mathrm{Cart}(G)$ is the image of the map $\mathbb{F}\mathsf{Proj}(kG) \to \mathbb{F}R_k(G)$ induced by the Cartan map $\mathsf{c}^G : \mathsf{Proj}(kG) \to R_k(G)$. We now show that the previous isomorphism is quite natural.

For this, we observe that the assignments $G \mapsto \mathsf{Proj}(kG)$ and $G \mapsto R_k(G)$ are diagonal p-permutation functors: If M is a diagonal p-permutation (kH, kG)-bimodule, then M is left and right projective, so if Λ is a projective kG-module, then $M \otimes_{kG} \Lambda$ is a projective kH-module. Similarly, the functor $M \otimes_{kG} -$ changes a short exact sequence of kG-modules into a short exact sequence of kH-modules.

Moreover, the Cartan maps c^G form a morphism of diagonal p-permutation functors

$$\mathsf{c} : \mathsf{Proj}(k-) \to R_k(-).$$

In particular the assignment $\mathbb{F}\mathrm{Cart}(-) : G \mapsto \mathbb{F}\mathrm{Cart}(G)$ is a subfunctor of $\mathbb{F}R_k(-)$.

Lemma 6.2: *The functor* $\mathbb{F}\mathrm{Cart}(-)$ *is the unique minimal subfunctor of* $\mathbb{F}R_k(-)$. *It is isomorphic to the simple functor* $\mathsf{S}_{\mathbf{1},1,\mathbb{F}}$.

Proof: Let F be a subfunctor of $\mathbb{F}R_k(-)$. Then $F(\mathbf{1}) \leq \mathbb{F}R_k(\mathbf{1}) = \mathbb{F}$, so $F(\mathbf{1})$ is either 0 or $\mathbb{F}$. Suppose first that $F(\mathbf{1}) = 0$. Let G be a finite group, and $u \in F(G)$. Then $u = \sum\limits_{S \in \mathrm{Irr}_k(G)} \lambda_S S$, where $\lambda_S \in \mathbb{F}$. Let $T \in \mathrm{Irr}_k(G)$, and P_T be its projective cover. Then $\mathsf{P}_T \in \mathbb{F}T^{\Delta}(\mathbf{1}, G)$, so $\mathsf{P}_T \otimes_{kG} u \in F(1) = 0$. But for $S \in \mathrm{Irr}_k(G)$, we have $\mathsf{P}_T \otimes_{kG} S = 0$ unless S is isomorphic to the dual $T^\natural$ of T. It follows that $\lambda_{T^\natural} = 0$ for any $T \in \mathrm{Irr}_k(G)$, so $u = 0$. Hence $F = 0$ if $F(\mathbf{1}) = 0$.

Suppose now that $F(\mathbf{1}) = \mathbb{F}$, that is $F(\mathbf{1}) \ni k$. If $T \in \mathrm{Irr}_k(G)$, then $\mathsf{P}_T \in \mathbb{F}T^{\Delta}(G, \mathbf{1})$, so $\mathsf{P}_T \otimes_k k \in F(G)$, for any $T \in \mathrm{Irr}_k(G)$. But $\mathsf{P}_T \otimes_k k \cong \mathsf{P}_T$ is the image of P_T by the Cartan map. It follows that $F(G)$ contains $\mathbb{F}\mathrm{Cart}(G)$, so $F \geq \mathbb{F}\mathrm{Cart}(-)$. Hence $\mathbb{F}\mathrm{Cart}(-)$ is the unique (non zero) minimal subfunctor of $\mathbb{F}R_k$. Since $\mathbb{F}\mathrm{Cart}(\mathbf{1}) \cong \mathbb{F}$, it follows that $\mathbb{F}\mathrm{Cart}(-) \cong \mathsf{S}_{\mathbf{1},1,\mathbb{F}}$, completing the proof. □

Remark 6.3: When $\mathbb{F}$ has characteristic p, the functor $\mathbb{F}\mathrm{Cart}(-)$ is a proper subfunctor of $\mathbb{F}R_k(-)$, so Lemma 6.2 shows in particular that the category $\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}$ is *not* semisimple.

6.4. In the case $\mathbb{F}$ has characteristic 0, the Cartan matrix has non zero determinant in $\mathbb{F}$, so the Cartan map $\mathbb{F}\mathsf{c}^G : \mathbb{F}\mathsf{Proj}(kG) \to \mathbb{F}R_k(G)$ is invertible. So we have isomorphisms of functors

$$\mathbb{F}\mathsf{Proj}(-) \cong \mathbb{F}R_k(-) \cong \mathsf{S}_{\mathbf{1},1,\mathbb{F}}$$

in this case. This is Theorem 5.20 in [6].

6.5. The other case we can consider is when $\mathbb{F}$ is a field of characteristic p, and we assume that $\mathbb{F} = k$. We choose a p-modular system $(K, \mathcal{O}, k)$, and we assume that K is big enough for the group G. If S is a simple kG-module, we denote by $\Phi_S : G_{p'} \to \mathcal{O}$ the modular character of P_S, where $G_{p'}$ is the set of p-regular elements of G. If $v = \sum\limits_{S \in \mathrm{Irr}_k(G)} \omega_S \mathsf{P}_S$, where $\omega_S \in \mathcal{O}$, is an element of $\mathcal{O}\mathsf{Proj}(kG)$, we denote by $\Phi_v : \mathcal{O}\mathsf{Proj}(kG) \to \mathcal{O}$ the map $\sum\limits_{S \in \mathrm{Irr}_k(G)} \omega_S \Phi_S : G_{p'} \to \mathcal{O}$, and we call Φ_v the modular character of v.

Then for a simple kG-module T, the multiplicity of S as a composition factor of P_T is equal to the Cartan coefficient

$$\mathsf{c}^G_{T,S} = \dim_k \mathrm{Hom}_{kG}(\mathsf{P}_T, \mathsf{P}_S) = \frac{1}{|G|} \sum_{x \in G_{p'}} \Phi_T(x)\Phi_S(x^{-1}).$$

In order to describe the image of the Cartan map $k\mathsf{c}^G$, we want to evaluate the image of this integer under the projection map $\rho : \mathcal{O} \to k$. For this, we denote by $[G_{p'}]$ a set of representatives of conjugacy classes of $G_{p'}$, and we observe that in the field K, we have

$$\begin{aligned}
\mathsf{c}^G_{S,T} &= \frac{1}{|G|} \sum_{x \in [G_{p'}]} \frac{|G|}{|C_G(x)|} \Phi_T(x)\Phi_S(x^{-1}) \\
&= \sum_{x \in [G_{p'}]} \frac{1}{|C_G(x)|} \frac{\Phi_T(x)}{|C_G(x)|_p} \frac{\Phi_S(x^{-1})}{|C_G(x)|_p} |C_G(x)|_p^2 \\
&= \sum_{x \in [G_{p'}]} \frac{\big(\Phi_T(x)/|C_G(x)|_p\big)\big(\Phi_S(x^{-1})/|C_G(x)|_p\big)}{|C_G(x)_{p'}|} |C_G(x)|_p. \qquad (6.5.1)
\end{aligned}$$

But since Φ_S and Φ_T are characters of projective kG-modules (and since $C_G(x) = C_G(x^{-1})$), the quotients $\Phi_T(x)/|C_G(x)|_p$ and $\Phi_S(x^{-1})/|C_G(x)|_p$ are in $\mathcal{O}$, so

$$\forall x \in [G_{p'}],\ \frac{\big(\Phi_T(x)/|C_G(x)|_p\big)\big(\Phi_S(x^{-1})/|C_G(x)|_p\big)}{|C_G(x)_{p'}|} \in \mathcal{O}.$$

Now it follows from 6.5.1 that

$$\rho(\mathsf{c}^G_{T,S}) = \sum_{x \in [G_0]} \rho\left(\frac{\Phi_T(x)\Phi_S(x^{-1})}{|C_G(x)|}\right), \qquad (6.5.2)$$

where $[G_0]$ is a set of representatives of conjugacy classes of the set G_0 of elements *defect zero* of G, i.e. the set of p'-elements x of G such that $C_G(x)$ is a p'-group.

Notation 6.6: *For $x \in G_0$, we set*

$$\Gamma_{G,x} = \sum_{S \in \mathrm{Irr}(kG)} \frac{\Phi_S(x^{-1})}{|C_G(x)|} S \in \mathcal{O}R_k(G),$$

where $\mathrm{Irr}(kG)$ *is a set of representatives of isomorphism classes of simple kG-modules. We also set*

$$\gamma_{G,x} = \sum_{S \in \mathrm{Irr}(kG)} \rho\left(\frac{\Phi_S(x^{-1})}{|C_G(x)|}\right) S \in kR_k(G),$$

Remark 6.7: We note that $\Gamma_{G,x}$ and $\gamma_{G,x}$ only depend on the conjugacy class of x in G, that is $\Gamma_{G,x} = \Gamma_{G,x^g}$ and $\gamma_{G,x} = \gamma_{G,x^g}$ for $g \in G$.

By Theorem 6.3.2 of [8] (see also Theorem 6.3.2 of [13]), the elementary divisors of the Cartan matrix of G are equal to $|C_G(x)|_p$, for $x \in [G_{p'}]$. It follows that the rank of the Cartan matrix modulo p, is equal to the number of conjugacy classes of elements of defect 0 of G, i.e. the cardinality of $[G_0]$. The following can be viewed as an explicit form of this result:

Proposition 6.8:

1. *Let T be a simple kG-module. Then, in $kR_k(G)$,*

$$k\mathsf{c}^G(\mathsf{P}_T) = \sum_{x \in [G_0]} \rho\big(\Phi_T(x)\big)\, \gamma_{G,x}.$$

2. *The elements $\gamma_{G,x}$, for $x \in [G_0]$, form a basis of $k\mathrm{Cart}(G) \leq kR_k(G)$.*

Proof: Throughout the proof, we simply write γ_x instead of $\gamma_{G,x}$.

1. By 6.5.2, we have

$$\begin{aligned}
k\mathsf{c}^G(\mathsf{P}_T) &= \sum_{S \in \mathrm{Irr}(kG)} \rho(\mathsf{c}^G_{T,S}) S = \sum_{S \in \mathrm{Irr}(kG)} \sum_{x \in [G_0]} \rho\left(\frac{\Phi_T(x)\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\
&= \sum_{x \in [G_0]} \sum_{S \in \mathrm{Irr}(kG)} \rho\left(\frac{\Phi_T(x)\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\
&= \sum_{x \in [G_0]} \rho\big(\Phi_T(x)\big) \sum_{S \in \mathrm{Irr}(kG)} \rho\left(\frac{\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\
&= \sum_{x \in [G_0]} \rho\big(\Phi_T(x)\big)\, \gamma_x.
\end{aligned}$$

2. We first prove that γ_x lies in the image of $k\mathsf{c}^G$, for any $x \in G_0$. So let $x \in G_0$, and $1_x : \langle x \rangle \to \mathcal{O}$ be the map with value 1 at x and 0 elsewhere. Then $|x|1_x = \sum_{\zeta} \zeta(x^{-1})\zeta$, where ζ runs through the simple $k\langle x \rangle$-modules, i.e. the group homomorphisms $\langle x \rangle \to \mathcal{O}^\times$, is an element of $\mathcal{O}P_k(\langle x \rangle) = \mathcal{O}R_k(\langle x \rangle)$. Let $v_x = \mathrm{Ind}_{\langle x \rangle}^G(|x|1_x)$. Then $v_x \in \mathcal{O}P_k(G)$, and its modular character evaluated at $g \in G$ is equal to

$$\begin{aligned}\Phi_{v_x}(g) &= \frac{1}{|x|} \sum_{\substack{h \in G \\ g^h \in \langle x \rangle}} \Phi_{|x|1_x}(g^h) \\ &= \frac{1}{|x|} \sum_{\substack{h \in G \\ g^h = x}} |x| = \begin{cases} |C_G(x)| & \text{if } g =_G x \\ 0 & \text{otherwise,} \end{cases}\end{aligned} \tag{6.8.1}$$

where $g =_G x$ means that g is conjugate to x in G. Now from Assertion 1, we get that

$$k\mathsf{c}^G(v_x) = \sum_{y \in [G_0]} \rho\big(\Phi_{v_x}(y)\big)\gamma_y = |C_G(x)|\gamma_x, \tag{6.8.2}$$

so γ_x is in the image of $k\mathsf{c}^G$, since $|C_G(x)| \neq 0$ in k.

Now by Assertion 1, the elements γ_x, for $x \in [G_0]$, generate the image $\mathrm{Cart}(G)$ of $k\mathsf{c}^G$. They are moreover linearly independent: Suppose indeed that some linear combination $\sum_{x \in [G_0]} \lambda_x \gamma_x$, where $\lambda_x \in k$, is equal to 0. For all $x \in [G_0]$, choose $\widetilde{\lambda}_x \in \mathcal{O}$ such that $\rho(\widetilde{\lambda}_x) = \lambda_x$. By (6.8.2), we get an element $\sum_{x \in [G_0]} \widetilde{\lambda}_x \frac{v_x}{|C_G(x)|}$ of $\mathcal{O}\mathsf{Proj}(kG)$ whose modular character has values in the maximal ideal $J(\mathcal{O})$ of $\mathcal{O}$. But by (6.8.1), the value at $g \in G_{p'}$ of this modular character is equal to

$$\sum_{x \in [G_0]} \widetilde{\lambda}_x \frac{\Phi_{v_x}(g)}{|C_G(x)|},$$

which is equal to 0 if $g \notin G_0$, and to $\widetilde{\lambda}_x$ if g is conjugate to $x \in [G_0]$ in G. It follows that $\widetilde{\lambda}_x \in J(\mathcal{O})$, hence $\lambda_x = \rho(\widetilde{\lambda}_x) = 0$. Since $g \in G_{p'}$ was arbitrary, we get that $\lambda_x = 0$ for any $x \in [G_0]$, so the elements γ_x, for $x \in [G_0]$, are linearly independent. This completes the proof of Proposition 6.8. □

6.9. The functors $\mathsf{S}_{L,1,W}$. In this section, we consider the case where $u = 1$, i.e. the case of simple functors $\mathsf{S}_{L,1,W}$, where L is a p-group and W is a simple $\mathbb{F}\mathrm{Out}(L)$ module. The case where $\mathbb{F}$ has characteristic 0 is solved by Corollary 7.4 of [7]. Then we are left with the case where $\mathbb{F}$ has characteristic p, and we assume that $\mathbb{F} = k$. In this situation, for a finite group G, the set $\mathcal{P}(G, L, 1)$ is just the set of pairs (Q, δ), where Q is a p-subgroup of G and $\delta : L \to Q$ is a group isomorphism. Moreover, the set $[\mathcal{P}(G, L, 1)]$ is in one to one correspondence with a set of representatives of conjugacy classes of subgroups Q of G which are isomorphic to L (the bijection mapping (Q,δ) to Q). Then the group $G_{Q,\delta,1}$ is just $N_G(Q)$, while $G_{Q,\delta} = QC_G(Q)$. By Theorem 5.25, we have that

$$\mathsf{S}_{L,1,W}(G) \cong \bigoplus_{(Q,\delta)\in[\mathcal{P}(G,L,1)]} \mathrm{Tr}_{\mathbf{1}}^{N_G(Q)/QC_G(Q)}\Big(k\mathrm{Cart}\big(C_G(Q)/Z(Q)\big)\otimes_k W\Big).$$

Notation 6.10: *Let G be a finite group.*

- *For a subgroup Q of G and an element z of G, we set $N_G(Q,z) = N_G(Q)\cap C_G(z)$ and $C_G(Q,z) = C_G(Q)\cap C_G(z)$*
- *For a p-subgroup Q of G, we denote by $\zeta(G,Q)$ the set of elements z in $C_G(Q)_{p'}$ for which $Z(Q)$ is a Sylow p-subgroup of $C_G(Q,z)$. The group $N_G(Q)$ acts on $\zeta(G,Q)$ by conjugation, and we denote by $[\zeta(G,Q)]$ a set of representatives of orbits under this action.*
- *For a finite p-group L, we denote by $\mathcal{Z}(G,L)$ the set of pairs (Q,z), where Q is a subgroup of G isomorphic to L, and z is an element of $\zeta(G,Q)$. In other words*

  $$\mathcal{Z}(G,L) = \big\{(Q,z)\mid L\cong Q\le G,\ z\in C_G(Q)_{p'},\ Z(Q)\in \mathrm{Syl}_p\big(C_G(Q,z)\big)\big\}.$$

  The group G acts on $\mathcal{Z}(G,L)$ by conjugation, and we denote by $[\mathcal{Z}(G,L)]$ a set of representatives of orbits under this action.

Theorem 6.11: *Let L be a p-group and W be a simple $k\mathrm{Out}(L)$-module. Let moreover G be a finite group.*

1. *Let Q be a p-subgroup of G. Then there is an isomorphism*

   $$k\mathrm{Cart}\big(C_G(Q)/Z(Q)\big) \cong \bigoplus_{z\in[\zeta(G,Q)]} \mathrm{Ind}_{N_G(Q,z)C_G(Q)/QC_G(Q)}^{N_G(Q)/QC_G(Q)} k$$

   of $kN_G(Q)/QC_G(Q)$-modules.

2. *The evaluation of the simple functor $\mathsf{S}_{L,1,W}$ at G is*

   $$\mathsf{S}_{L,1,W}(G) \cong \bigoplus_{(Q,z)\in[\mathcal{Z}(G,L)]} \mathrm{Tr}_{\mathbf{1}}^{N_G(Q,z)/QC_G(Q,z)}(W).$$

Proof: 1. For a subgroup Q of G, denote by $x \mapsto \overline{x}$ the projection map $N_G(Q) \to \overline{N}_G(Q) = N_G(Q)/Q$, and set $\overline{C}_G(Q) = QC_G(Q)/Q \cong C_G(Q)/Z(Q)$.

By Proposition 6.8, the vector space $k\mathrm{Cart}\big(\overline{C}_G(Q)\big)$ has a basis consisting of the elements $\gamma_{\overline{C}_G(Q),x}$, for $x \in [\overline{C}_G(Q)_0]$, and the group $N_G(Q)$ permutes these elements. Now if $x \in \overline{C}_G(Q)_{p'}$, there is an element $z \in \big(QC_G(Q)\big)_{p'}$ such that $x = \overline{z}$, and we can moreover assume that $z \in C_G(Q)_{p'}$. Then the centralizer of $\overline{z}$ in $\overline{C}_G(Q)$ is equal to $QC_{QC_G(Q)}(z)/Q = QC_G(Q,z)/Q$. So $\overline{z} \in \overline{C}_G(Q)_0$ if and only if Q is a Sylow p-subgroup of $QC_G(Q,z)$, or equivalently, if $Z(Q)$ is a Sylow p-subgroup of $C_G(Q,z)$, that is $z \in \zeta(G,Q)$.

Moreover, an element $n \in N_G(Q)$ stabilizes $\gamma_{\overline{C}_G(Q),\overline{z}}$ if and only if $\gamma_{\overline{C}_G(Q),\overline{z}} = \gamma_{\overline{C}_G(Q),\overline{nzn^{-1}}}$, hence by Proposition 6.8, if there exists $\overline{c} \in \overline{C}_G(Q)$ such that $\overline{nzn^{-1}} = \overline{czc}^{-1}$. In other words $\overline{c^{-1}n} \in C_{\overline{N}_G(Q)}(\overline{z}) = N_G(Q,z)/Q$, where we set $N_G(Q,z) = N_G(Q) \cap C_G(z)$. So the stabilizer of $\gamma_{\overline{C}_G(Q),\overline{z}}$ in $N_G(Q)$ is equal to $QC_G(Q)N_G(Q,z) = N_G(Q,z)C_G(Q)$.

Hence $k\mathrm{Cart}\big(\overline{C}_G(Q)\big)$ is isomorphic to the permutation $N_G(Q)/QC_G(Q)$-module $k\zeta(G,Q)$. The elements $\gamma_{\overline{C}_G(Q),\overline{z}}$, for $z \in [\zeta(G,Q)]$ form a set of representatives of orbits for the action of $N_G(Q)/QC_G(Q)$, and the stabilizer of $\gamma_{\overline{C}_G(Q),\overline{z}}$ is the group $N_G(Q,z)C_G(Q)/QC_G(Q)$. This proves Assertion 1.

2. Now $N_G(Q,z)C_G(Q)/QC_G(Q) \cong N_G(Q,z)/QC_G(Q,z)$, and Assertion 2 follows from Theorem 5.25, thanks to the general following fact (see Proposition 5.6.10 (ii) in [1]): If Γ' is a subgroup of a finite group Γ, if M is a finite dimensional $k\Gamma$-module and M' is a finite dimensional $k\Gamma'$-module, then

$$\mathrm{Tr}_{\mathbf{1}}^{\Gamma}\big((\mathrm{Ind}_{\Gamma'}^{\Gamma} M') \otimes_k M\big) \cong M' \otimes_k \mathrm{Tr}_{\mathbf{1}}^{\Gamma'}(\mathrm{Res}_{\Gamma'}^{\Gamma} M)$$

as k-vector spaces. □

Remark 6.12: The formula in Assertion 2 of Theorem 6.11 can be viewed as another instance of similar formulas in Proposition 8.8 of [14], Theorem 2.6 of [15], or Theorem 6.1 of [4].

The following corollary deals with the case of Theorem 6.11 where W is the trivial module k. First a definition:

Definition 6.13: *Let G be a finite group, and L be a finite p-group. An element $z \in G_{p'}$ is said to have* defect isomorphic to L *if L is isomorphic to a Sylow p-subgroup of $C_G(z)$.*

Corollary 6.14: *Let G be a finite group, and L be a finite p-group. Then the dimension of $\mathsf{S}_{L,1,k}(G)$ is equal to the number of conjugacy classes of elements of $G_{p'}$ with defect isomorphic to L.*

Proof: Indeed, if $(Q,z) \in \mathcal{Z}(G,L)$ then $\mathrm{Tr}_{\mathbf{1}}^{N_G(Q,z)/QC_G(Q)}(k)$ is equal to zero if p divides the order of $N_G(Q,z)/QC_G(Q)$, and one dimensional otherwise, that is, if a Sylow p-subgroup of $N_G(Q,z)$ is contained in $QC_G(Q,z)$. But since $z \in \zeta(G,Q)$, the group Q is a Sylow p-subgroup of $QC_G(Q,z)$. So $\mathrm{Tr}_{\mathbf{1}}^{N_G(Q,z)/QC_G(Q)}(k)$ is non zero (and then, one dimensional) if and only if Q is a Sylow p-subgroup of $N_G(Q,z) = N_{C_G(z)}(Q)$, i.e. if Q is a Sylow p-subgroup of $C_G(z)$. □

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