p-Bifree biset functors

Olcay COKUN[1] and Deniz YILMAZ[2]

[1] Center for Mathematical Research, ASOIU, Baku, Azerbaijan
olcay.coshkun@asoiu.edu.az
[2] Bilkent University, Ankara, Turkey
d.yilmaz@bilkent.edu.tr

Abstract

We introduce and study the category of p-bifree biset functors for a fixed prime p, defined via bisets whose left and right stabilizers are p'-groups. This category naturally lies between the classical biset functors and the diagonal p-permutation functors, serving as a bridge between them. Every biset functor and every diagonal p-permutation functor restricts to a p-bifree biset functor.

We classify the simple p-bifree biset functors over a field K of characteristic zero, showing that they are parametrized by pairs (G, V), where G is a finite group and V is a simple $K\mathrm{Out}(G)$-module. As key examples, we compute the composition factors of several representation-theoretic functors in the p-bifree setting, including the Burnside ring functor, the p-bifree Burnside functor, the Brauer character ring functor, and the ordinary character ring functor. We further investigate the classical simple biset functors $S_{G,\mathbb{C}}$ where G is a B-group.

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1 Introduction

The theory of biset functors, which was introduced and extensively developed by Bouc, occupies a central position in the functorial representation theory of finite groups. It enables a unified treatment of representation rings when the structural maps restriction, induction, deflation, inflation, and isogation are present. The completion of the classification of endo-permutation modules of p-groups [Bc06] and the description of the unit group of Burnside rings of p-groups [Bc07], both due to Bouc, are two notable applications of the theory of biset functors.

Diagonal p-permutation functors, introduced by Bouc and the second author [BY20], provide a functorial framework for studying structures involving actions of p-permutation bimodules with additional constraints. By replacing bisets with p-permutation bimodules whose vertices are twisted diagonals, this theory captures essential representation-theoretic phenomena, particularly those related to block theory. Diagonal p-permutation functors have already found applications in the block theory of finite groups; see, for instance, the finiteness result in terms of functorial equivalences, Theorem 10.6 in [BY22], which is in the spirit of Puig's and Donovan's finiteness conjectures.

Although the two theories of biset functors and diagonal p-permutation functors are both defined on categories of finite groups, there is no direct functorial connection between them. However, their morphisms are related as follows. A diagonal p-permutation bimodule is a p-permutation bimodule whose indecomposable summands have (twisted) diagonal subgroups as vertices. The linearization map applied to bisets yields permutation, and hence p-permutation bimodules. In particular, the elementary bisets

$$\mathrm{Res}^G_H \;\; (H \leqslant G), \;\; \mathrm{Ind}^G_H \;\; (H \leqslant G), \;\; \mathrm{Iso}(f) \;\; (f : G \xrightarrow{\sim} H),$$

$$\mathrm{Inf}^G_{G/N}, \mathrm{Def}^G_{G/N} \;\; (N \trianglelefteq G, \; N \text{ a normal } p'\text{-subgroup})$$

give rise to diagonal p-permutation bimodules via linearization. In contrast, inflations and deflations along general normal subgroups do not yield diagonal p-permutation bimodules under linearization; see [BY20, Lemma 4.2]. This obstruction prevents the existence of a direct functor between the categories of biset functors and diagonal p-permutation functors.

Motivated by this observation, we introduce the notions of p-bifree bisets and p-bifree biset functors. In fact, these are special cases of general notions introduced by Bouc [Bc10, Section 4.1.9] and by Webb [W00, Section 8]. A *p-bifree biset* is a biset with p'-stabilizers on both sides. The category of p-bifree bisets is similar to the classical biset category, but only includes inflations and deflations via normal p'-subgroups. For a commutative ring R with unity, we denote by $R\mathcal{C}^{\Delta,p}$ the category whose objects are finite groups and whose morphisms are given by the R-linear extension of the Grothendieck group $RB^{\Delta,p}(H,G)$ of p-bifree (H,G)-bisets. An R-linear functor from $R\mathcal{C}^{\Delta,p}$ to the category of R-modules is called a *p-bifree biset functor over R*.

With this definition, the category of p-bifree biset functors lies naturally between classical biset functors and diagonal p-permutation functors. On one hand, it contains all classical biset functors via restriction to the p-bifree part of the biset category. On the other hand, diagonal p-permutation functors factor through it via the linearization map, since only bisets with p'-stabilizers induce diagonal p-permutation bimodules under linearization. In this sense, p-bifree biset functors form a common generalization of both theories, effectively building a bridge between them.

Using Bouc's theory on linear functors [Bc96], we show that the simple p-bifree biset functors $S^{\Delta,p}_{G,V}$ over R are parametrized by the pairs (G,V) of a finite group G and a simple $R\mathrm{Out}(G)$-module V. We also consider various representation rings as p-bifree biset functors and determine their composition factors. Our main results can be summarized as follows.

- Let $\mathbb{K}$ be a field of characteristic zero. We prove that (Corollary 4.9) the composition factors of the Burnside ring as the p-bifree biset functor over $\mathbb{K}$ are exactly the functors $S_{G,\mathbb{K}}^{\Delta,p}$ where G is a $B^{\Delta,p}$-group. See Definition 4.3 for the definition of $B^{\Delta,p}$-group. The proof follows [Bc10, Chapter 5] very closely.

- The composition factors of the representable p-bifree biset functor $\mathbb{K}B^{\Delta,p}(-,1)$ are exactly the functors $S_{G,\mathbb{K}}^{\Delta,p}$ where G is a B-group of p'-order (Corollary 5.4).

- Let G be a finite group. The $\mathbb{K}$-dimension of $S_{1,\mathbb{K}}^{\Delta,p}(G)$ is equal to the number of conjugacy classes of cyclic p'-subgroups of G (Theorem 6.1).

- We also investigate the p-bifree biset functor structures of the ordinary and Brauer character rings of finite groups. Let $\mathbb{C}$ be an algebraically closed field of characteristic zero and let k be an algebraically closed field of characteristic $p > 0$. Let $R_{\mathbb{C}}(G)$ and $R_k(G)$ denote the Grothendieck groups of $\mathbb{C}G$ and kG-modules, respectively. We show that both $\mathbb{C}R_{\mathbb{C}}(-)$ and $\mathbb{C}R_k(-)$ are semisimple p-bifree biset functors. One has

$$\mathbb{C}R_{\mathbb{C}} \cong \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}^{\Delta,p}$$

where (m,ξ) runs through the set of pairs consisting of a positive integer m and a p-primitive character (see Definition 7.4) ξ of $(\mathbb{Z}/m\mathbb{Z})^\times$ (Corollary 7.7). Furthermore,

$$\mathbb{C}R_k \cong \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}^{\Delta,p}$$

where (m,ξ) runs through a set of pairs consisting of a positive p'-integer m and a primitive character ξ of $(\mathbb{Z}/m\mathbb{Z})^\times$ (Corollary 8.4). As a result of these we obtain a short exact sequence

$$0 \longrightarrow \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}^{\Delta,p} \longrightarrow \mathbb{C}R_{\mathbb{C}} \longrightarrow \mathbb{C}R_k \longrightarrow 0$$

of p-bifree biset functors, where (m,ξ) runs through the set of pairs consisting of a positive integer m divisible by p and a p-primitive character ξ of $(\mathbb{Z}/m\mathbb{Z})^\times$.

- Finally, we consider some simple biset functors as p-bifree biset functors. Let $S_{G,V}$ denote the simple biset functor associated to the pair (G,V) of a finite group G and a simple $\mathbb{C}\mathrm{Out}(G)$-module V. Then, if K is a B-group, we have an isomorphism

$$S_{K,\mathbb{C}} \cong \bigoplus_{H} S_{H,\mathbb{C}}^{\Delta,p}$$

of p-bifree biset functors, where H runs over a set of isomorphism classes of $B^{\Delta,p}$-groups with the property $\beta(H) \cong K$ (Theorem 9.2).

2 Category of p-bifree bisets

Let $p > 0$ be a prime and let G, H and K denote finite groups.

We denote by $p_1 : H \times G \to H$ and $p_2 : H \times G \to G$ the canonical projections. For a subgroup $L \leqslant H \times G$, we set

$$k_1(L) = \{h \in H \mid (h,1) \in L\} \quad \text{and} \quad k_2(L) = \{g \in G \mid (1,g) \in L\}\,.$$

One has canonical isomorphisms

$$q(L) := L/(k_1(L) \times k_2(L)) \to p_i(L)/k_i(L)$$

for $i = 1, 2$, induced by the projections p_i.

2.1 Definition Let X be an (H, G)-biset. We say X is a *p-bifree (H, G)-biset* if the left and right stabilizers of the $H \times G$-orbits of X are p'-groups. In other words, X is a disjoint union of transitive (H, G)-bisets of the form $[(H \times G)/L]$ where $k_1(L)$ and $k_2(L)$ are p'-groups.

2.2 (a) Let ${}_H\mathsf{set}_G^{\Delta,p}$ denote the category of p-bifree (H, G)-bisets. Using the Mackey formula, one shows that the composition of bisets induces a bilinear map

$${}_K\mathsf{set}_H^{\Delta,p} \times {}_H\mathsf{set}_G^{\Delta,p} \to {}_K\mathsf{set}_G^{\Delta,p}.$$

Let $B^{\Delta,p}(H, G)$ denote the Grothendieck group of the category ${}_H\mathsf{set}_G^{\Delta,p}$. The product of bisets induces a well-defined map

$$- \cdot_H - : B^{\Delta,p}(K, H) \times B^{\Delta,p}(H, G) \to B^{\Delta,p}(K, G)\,. \tag{1}$$

Recall from [Bc10] that every transitive (H, G)-biset can be written as a product of five elementary bisets:

$$\left[\frac{H \times G}{L}\right] = \mathrm{Ind}_{p_1(L)}^{H} \circ \mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} \circ \mathrm{Iso}(f) \circ \mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)} \circ \mathrm{Res}_{p_2(L)}^{G}\,,$$

where $f : p_2(L)/k_2(L) \to p_1(L)/k_1(L)$ is the canonical isomorphism. One has $\mathrm{Inf}_{G/N}^{G} \in B^{\Delta,p}(G, G/N)$ if and only if N is a p'-subgroup of G. Similarly, $\mathrm{Def}_{G/N}^{G} \in B^{\Delta,p}(G/N, G)$ if and only if N is a p'-subgroup of G. Hence $B^{\Delta,p}(H, G)$ is generated by bisets of the above form for $k_1(L)$ and $k_2(L)$ p'-groups.

(b) Let R be a commutative ring with 1. Let $R\mathcal{C}^{\Delta,p}$ denote the following category:

- objects: finite groups,
- $\mathrm{Hom}_{R\mathcal{C}^{\Delta,p}}(G, H) = R \otimes_{\mathbb{Z}} B^{\Delta,p}(H, G) = RB^{\Delta,p}(H, G)$,

- composition is induced from the map in (1),
- $\mathrm{Id}_G = [\mathrm{Id}_G] = [G]$.

(c) Let G be a finite group. We set

$$I_G = \sum_{|H|<|G|} RB^{\Delta,p}(G,H) \circ RB^{\Delta,p}(H,G)\,.$$

Note that I_G is an ideal of the endomorphism ring $\mathrm{End}_{R\mathcal{C}^{\Delta,p}}(G) = RB^{\Delta,p}(G,G)$. The quotient

$$\mathcal{E}^{\Delta,p}(G) := RB^{\Delta,p}(G,G)/I_G$$

is called the *essential algebra* of G in $R\mathcal{C}^{\Delta,p}$.

A transitive p-bifree (G,G)-biset $(G\times G)/L$ has zero image in $\mathcal{E}^{\Delta,p}(G)$ if and only if $|q(L)| < |G|$. Hence as in the usual biset category, see [Bc10, Proposition 4.3.2], we have an isomorphism

$$\mathcal{E}^{\Delta,p}(G) \cong R\mathrm{Out}(G)\,.$$

3 p-bifree biset functors

In this section, we classify the simple objects in the category $\mathcal{F}_R^{\Delta,p}$ of p-bifree biset functors over a commutative ring R. Our construction follows the general theory of linear biset functors developed by Bouc in [Bc96].

3.1 Definition An R-linear functor $R\mathcal{C}^{\Delta,p} \to {}_R\mathsf{Mod}$ is called a *p-bifree biset functor over R*.

Together with natural transformations, p-bifree biset functors over R form an abelian category which we denote by $\mathcal{F}_R^{\Delta,p}$.

By [Bc10, Theorem 4.3.10], the simple p-bifree biset functors over R are parametrized by the pairs (G,V) where G is a finite group and V is a simple $R\mathrm{Out}(G)$-module. For a given pair (G,V) we denote the corresponding simple p-bifree biset functor by $S_{G,V}^{\Delta,p}$.

3.2 Remark (a) Every biset functor becomes a p-bifree biset functor via the restriction along the inclusion $R\mathcal{C}^{\Delta,p} \subseteq R\mathcal{C}$, where $R\mathcal{C}$ denotes the usual biset category over R.

(b) Let Rpp_k^{Δ} denote the diagonal p-permutation category over R, see [BY20, Definition 2.6], or [BY22, Section 1]. The linearization map $X \mapsto kX$ induces a functor $R\mathcal{C}^{\Delta,p} \to Rpp_k^{\Delta}$. This makes every diagonal p-permutation functor over R a p-bifree biset functor.

3.3 Definition Let F be a p-bifree biset functor over R. A simple p-bifree biset functor S is called a *composition factor* of F, if there exist subfunctors $F_2 < F_1 \leq F$ such that $F_1/F_2 \cong S$.

Note that if F is a p-bifree biset functor over a field, then F admits a composition factor. This follows since the proof of [Bc18, Lemma 9.7(i)] applies verbatim to p-bifree biset functors; see also the remarks at the beginning of [BST13, Section 3].

3.4 Definition Let F be a p-bifree biset functor and let G be a finite group. We define the submodules $\underline{F}(G)$ and $\mathcal{J}F(G)$ of $F(G)$ by

$$\underline{F}(G) = \bigcap_{\substack{|H|<|G| \\ \alpha \in RB^{\Delta,p}(H,G)}} \mathrm{Ker}(F(\alpha) : F(G) \to F(H))$$

and

$$\mathcal{J}F(G) = \sum_{\substack{|H|<|G| \\ \alpha \in RB^{\Delta,p}(G,H)}} \mathrm{Im}(F(\alpha) : F(H) \to F(G))\,.$$

Notice that both $\underline{F}(-)$ and $\mathcal{J}F(-)$ are subfunctors of F. The subfunctor $\underline{F}(-)$ is called the *restriction kernel* of F. These restriction kernels serve as a tool for detecting minimal groups and composition factors. They will be used repeatedly in later sections to describe the subfunctor structure and to prove that certain families of simple functors exhaust all composition factors. See Appendix for more properties of the restriction kernels.

4 Burnside ring as a p-bifree biset functor

Let R be a commutative ring with identity and let $\mathbb{K}$ be a field of characteristic zero. The *Burnside functor* $RB : R\mathcal{C}^{\Delta,p} \to {}_R\mathsf{Mod}$ is defined as the restriction of the Burnside biset functor RB to the p-bifree biset category, that is,

- $G \mapsto RB(G) := R \otimes_{\mathbb{Z}} B(G)$.
- $X \in RB^{\Delta,p}(H,G) \mapsto (RB(X) : RB(G) \to RB(H), \quad U \mapsto X \cdot_G U)$.

In this section we analyze the structure of RB. It turns out that the results are very similar to those in Chapter 5 of [Bc10], where the structure of the Burnside functor is described in terms of B-groups. In our setting, the same arguments apply almost line by line once one restricts to inflations and deflations along normal p'-subgroups. We include the full details here for completeness and to set up the theory in the context of p-bifree biset functors. The notion of a $B^{\Delta,p}$-group plays the role of a B-group in this setting, and many structural results, including the parametrization of composition factors and the poset of subfunctors, carry over with suitable modifications. The first and key result is the following characterization of evaluations of subfunctors. We refer to [Bc10, Lemma 5.2.1] for its proof.

4.1 Lemma *Let F be a p-bifree biset subfunctor of RB. Then for any finite group G, the R-module $F(G)$ is an ideal of $RB(G)$.*

Note that $\mathbb{K}B(G)$ is a split semisimple $\mathbb{K}$-algebra. In particular, every ideal of $\mathbb{K}B(G)$ is generated by a set of primitive idempotents.

The primitive idempotents e_H^G of $\mathbb{K}B(G)$ are indexed by the conjugacy classes $[s_G]$ of subgroups $H \leqslant G$. Let $\mathbf{e}_G^{\Delta,p}$ denote the p-bifree biset subfunctor of $\mathbb{K}B$ generated by $e_G^G \in \mathbb{K}B(G)$. We note that this is in general smaller than the biset subfunctor $\mathbf{e}_G$ of $\mathbb{K}B$ generated by e_G^G. Recall from [Bc10, Theorem 5.2.4] that for a normal subgroup N of G we have

$$\mathrm{Def}^G_{G/N} e^G_G = m_{G,N} e^{G/N}_{G/N} \,,$$

where the deflation number $m_{G,N}$ is given by

$$m_{G,N} = \frac{1}{|G|} \sum_{XN=G} |X| \mu(X,G) \,.$$

Recall from [Bc10, Definition 2.3.12] that a *section* (Y,X) of a finite group G is a pair of subgroups of G with $X \trianglelefteq Y$. The quotient group Y/X is called the *subquotient* associated with (Y,X).

4.2 Proposition *Let G be a finite group. The following conditions are equivalent.*

(i) *The evaluation of the subfunctor $\mathbf{e}_G^{\Delta,p}$ vanishes on all groups of smaller order, that is, $\mathbf{e}_G^{\Delta,p}(H) = \{0\}$ for all $|H| < |G|$.*

(ii) *If $\mathbf{e}_G^{\Delta,p}(H) \neq \{0\}$ for some finite group H, then G is isomorphic to a subquotient Y/X associated with a section (Y,X) of H with X a p'-group.*

(iii) *The deflation number $m_{G,N}$ vanishes for every non-trivial normal p'-subgroup $N \trianglelefteq G$.*

(iv) *The deflation $\mathrm{Def}^G_{G/N} e^G_G$ is zero in $\mathbb{K}B(G/N)$ for every non-trivial normal p'-subgroup $N \trianglelefteq G$.*

Proof The equivalence (iii) $\iff$ (iv) follows directly from Theorem 5.2.4 in [Bc10]. Suppose (i) holds. Let $N \trianglelefteq G$ be a non-trivial p'-subgroup. Then the image of e_G^G under deflation lies in $\mathbf{e}_G^{\Delta,p}(G/N) = \{0\}$, so $\mathrm{Def}^G_{G/N} e^G_G = 0$, and hence (iv) holds. Now suppose (iv) holds and let H be a finite group such that $\mathbf{e}_G^{\Delta,p}(H) \neq \{0\}$. Then there exists $\varphi \in \mathbb{K}B^{\Delta,p}(H,G)$ such that $\varphi(e_G^G) \neq 0$. In particular there exist sections (Y,X) of H and (T,S) of G, with X and S p'-groups, and an isomorphism $f : T/S \to Y/X$ such that

$$\mathrm{Ind}_Y^H \mathrm{Inf}^Y_{Y/X} \mathrm{Iso}(f) \mathrm{Def}^T_{T/S} \mathrm{Res}^G_T (e^G_G) \neq 0.$$

By [Bc10, Theorem 5.2.4], the restriction of e_G^G to a proper subgroup is zero, hence $T = G$. Condition (iv) further implies that $S = 1$. This proves (ii). Finally, (ii) $\Rightarrow$ (i) is immediate. □

4.3 Definition A finite group G is called a $B^{\Delta,p}$*-group* if for any non-trivial normal p'-subgroup N of G, the deflation number $m_{G,N}$ is equal to zero.

4.4 Remark When the condition on N is removed, we recover Bouc's definition of a B-group. These two definitions can be compared as follows:

(i) Every B-group is a $B^{\Delta,p}$-group.

(ii) Every p-group is a $B^{\Delta,p}$-group.

(iii) A p-group which is also a B-group is either trivial or elementary abelian of rank 2, by 5.6.9 of [Bc10].

(iv) A p'-group is a $B^{\Delta,p}$-group if and only if it is a B-group.

4.5 Theorem *Let G and H be finite groups. Then the following hold:*

(i) *If H is isomorphic to a quotient of G by a p'-subgroup, then $\mathbf{e}_G^{\Delta,p} \subseteq \mathbf{e}_H^{\Delta,p}$.*

(ii) *If H is a $B^{\Delta,p}$-group and $\mathbf{e}_G^{\Delta,p} \subseteq \mathbf{e}_H^{\Delta,p}$, then H is isomorphic to a quotient of G by a p'-subgroup.*

(iii) *If F is a subfunctor of $\mathbb{K}B$ and H is a minimal group of F, then H is a $B^{\Delta,p}$-group, $F(H) = \mathbb{K}e_H^H$, and $\mathbf{e}_H^{\Delta,p} \subseteq F$. In particular, $\mathbf{e}_H^{\Delta,p}(H) = \mathbb{K}e_H^H$ if H is a $B^{\Delta,p}$-group.*

(iv) *The minimal group $\delta_p(G)$ of $\mathbf{e}_G^{\Delta,p}$ is uniquely determined up to isomorphism. One has $\mathbf{e}_G^{\Delta,p} = \mathbf{e}_{\delta_p(G)}^{\Delta,p}$, and $\delta_pG)$ is isomorphic to a quotient of G by a normal p'-subgroup. Moreover, for any normal p'-subgroup $N \trianglelefteq G$ such that $G/N \cong \delta_p(G)$, one has $m_{G,N} \neq 0$.*

(v) *Let $\beta(G)$ be a minimal group of the biset subfunctor $\mathbf{e}_G$ of $\mathbb{K}B$ generated by the idempotent e_G^G. Then $\beta(\delta_p(G)) \cong \beta(G)$.*

Proof (i) Let $N \trianglelefteq G$ be a p'-subgroup such that $G/N \cong H$ via an isomorphism f. Then, by [Bc10, Theorem 5.2.4], one has

$$e_G^G \mathrm{Inf}_{G/N}^G \mathrm{Iso}(f) e_H^H = e_G^G,$$

so $e_G^G \in \mathbf{e}_H^{\Delta,p}(G)$ and hence $\mathbf{e}_G^{\Delta,p} \subseteq \mathbf{e}_H^{\Delta,p}$.

(ii) Since $e_G^G \in \mathbf{e}_H^{\Delta,p}(G) = \mathbb{K}B^{\Delta,p}(G,H)e_H^H$, there exist sections (T,S) of G and (Y,X) of H with S and X p'-groups and an isomorphism $f : Y/X \to T/S$ such that

$$e_G^G \mathrm{Ind}_T^G \mathrm{Inf}_{T/S}^T \mathrm{Iso}(f) \mathrm{Def}_{Y/X}^Y \mathrm{Res}_Y^H e_H^H \neq 0.$$

As in previous arguments, since the restriction of e_H^H to a proper subgroup is zero, we have $Y = H$, and since H is a $B^{\Delta,p}$-group, we get $X = 1$. Also because $e_G^G[G/L] = 0$ for any proper subgroup L, we have $T = G$ and the result follows.

(iii) This follows from the proof of Proposition 5.4.9 in [Bc10] applied in the p-bifree setting. The minimality of H implies that $F(H)$ is generated by e_H^H, and hence $\mathbf{e}_H^{\Delta,p} \subseteq F$.

(iv) This is a direct adaptation of Proposition 5.4.10 in [Bc10], replacing all normal subgroups with normal p'-subgroups.

(v) First note that the group $\beta(G)$ is uniquely determined, up to isomorphism, by [Bc10, Proposition 5.4.10]. Let $N \trianglelefteq G$ be a p'-normal subgroup with $G/N \cong \delta_p(G)$. By Part (iv), we have $m_{G,N} \neq 0$ which is equivalent to $\beta(\delta_p(G)) \cong \beta(G/N) \cong \beta(G)$ by [Bc10, Theorem 5.4.11]. □

The following theorem collects properties of the minimal group $\delta_p(G)$ of the subfunctor $\mathbf{e}_G^{\Delta,p}$. As in the classical case, the key results follow from Theorem 5.4.11 in [Bc10], and the same arguments apply here with the necessary modifications to the p-bifree setting.

4.6 Theorem *Let G be a finite group.*

(i) Let H be a $B^{\Delta,p}$-group that is isomorphic to a quotient of G by a normal p'-subgroup. Then H is also isomorphic to a quotient of $\delta_p(G)$ by a normal p'-subgroup.

(ii) Let $N \trianglelefteq G$ be a normal p'-subgroup. The following are equivalent:

- *(a) $m_{G,N} \neq 0$,*
- *(b) $\delta_p(G)$ is isomorphic to a quotient of G/N by a p'-subgroup,*
- *(c) $\delta_p(G) \cong \delta_p(G/N)$.*

(iii) In particular, if $N \trianglelefteq G$ is a normal p'-subgroup, then $G/N \cong \delta_p(G)$ if and only if G/N is a $B^{\Delta,p}$-group and $m_{G,N} \neq 0$.

Let $B^{\Delta,p}$-gr denote the class of $B^{\Delta,p}$-groups, and let $[B^{\Delta,p}\text{-gr}]$ be a fixed set of representatives of their isomorphism classes. Define a relation $\gg$ on $B^{\Delta,p}$-gr by declaring $G \gg H$ if and only if H is isomorphic to a quotient of G by a p'-group. A subset $\mathcal{M} \subseteq B^{\Delta,p}$-gr is said to be *closed* if, whenever $H \in \mathcal{M}$ and $G \gg H$, it follows that $G \in \mathcal{M}$.

The following result describes the lattice of subfunctors of the Burnside functor $\mathbb{K}B$ in terms of these closed subsets. Its proof is the same as that of Theorem 5.4.14 in [Bc10], with the necessary modifications to the p-bifree bisets.

4.7 Theorem *Let $\mathcal{S}$ be the set of p-bifree biset subfunctors of the Burnside functor $\mathbb{K}B$, ordered by inclusion, and let $\mathcal{T}$ be the set of closed subsets of the set $[B^{\Delta,p}\text{-gr}]$ of isomorphism classes of $B^{\Delta,p}$-groups, ordered by inclusion. Then the assignment*

$$\Theta : F \mapsto \{H \in [B^{\Delta,p}\text{-gr}] \mid \mathbf{e}_H^{\Delta,p} \subseteq F\}$$

defines an isomorphism of posets $\Theta : \mathcal{S} \xrightarrow{\sim} \mathcal{T}$.

Its inverse is given by

$$\Psi : A \mapsto \sum_{H \in A} \mathbf{e}_H^{\Delta,p}.$$

We now describe the composition factors of the Burnside functor $\mathbb{K}B$ as a p-bifree biset functor. For each $B^{\Delta,p}$-group G, the subfunctor $\mathbf{e}_G^{\Delta,p}$ is crucial, and its maximal proper subfunctor determines a unique simple quotient. We refer to Section 5 of [Bc10] for the proofs of the following results up to the end of the section.

4.8 Proposition (i) *Let G be a $B^{\Delta,p}$-group. Then the functor $\mathbf{e}_G^{\Delta,p}$ has a unique maximal subfunctor*

$$\mathbf{j}_G^{\Delta,p} = \sum_{\substack{H\in[B^{\Delta,p}\text{-gr}]\\ H\gg G,\ H\ncong G}} \mathbf{e}_H^{\Delta,p},$$

and the quotient $\mathbf{e}_G^{\Delta,p}/\mathbf{j}_G^{\Delta,p}$ is isomorphic to the simple functor $S_{G,\mathbb{K}}^{\Delta,p}$.

(ii) *Let $F \subset F'$ be subfunctors of $\mathbb{K}B$ such that F'/F is simple. Then there exists a unique $G \in [B^{\Delta,p}\text{-gr}]$ such that $\mathbf{e}_G^{\Delta,p} \subseteq F'$ and $\mathbf{e}_G^{\Delta,p} \nsubseteq F$. In particular, one has*

$$\mathbf{e}_G^{\Delta,p} + F = F', \quad \mathbf{e}_G^{\Delta,p} \cap F = \mathbf{j}_G^{\Delta,p}, \quad \text{and} \quad F'/F \cong S_{G,\mathbb{K}}^{\Delta,p}.$$

4.9 Corollary *The composition factors of $\mathbb{K}B$ as a p-bifree biset functor are precisely the simple functors $S_{G,\mathbb{K}}^{\Delta,p}$, where G runs over the set $[B^{\Delta,p}\text{-gr}]$.*

To compute the evaluation of the simple functor $S_{G,\mathbb{K}}^{\Delta}$ at a finite group H, we first describe the structure of $\mathbf{e}_G^{\Delta,p}(H)$ in terms of the subgroup lattice of H.

4.10 Proposition *Let G and H be finite groups. Then the evaluation $\mathbf{e}_G^{\Delta,p}(H)$ is the subspace of $\mathbb{K}B(H)$ spanned by the idempotents e_K^H, where K runs over a set of representatives of conjugacy classes of subgroups of H satisfying $K \gg \delta_p(G)$.*

This description allows us to compute the dimension of $S_{G,\mathbb{K}}^{\Delta,p}(H)$ explicitly in terms of the minimal groups $\delta_p(K)$ associated with the subgroups $K \leqslant H$.

4.11 Theorem *Let G be a $B^{\Delta,p}$-group and H a finite group. Then the $\mathbb{K}$-dimension of $S_{G,\mathbb{K}}^{\Delta,p}(H)$ is equal to the number of conjugacy classes of subgroups $K \leqslant H$ such that $\delta_p(K) \cong G$.*

5 The functor $\mathbb{K}B^{\Delta,p}$

In this section, we study the representable functor $\mathbb{K}B^{\Delta,p}(-,1)$, which assigns to each finite group G the $\mathbb{K}$-vector space $\mathbb{K}B^{\Delta,p}(G,1)$. This functor is a p-bifree analogue of the classical representable biset functor $\mathbb{K}B(-,1)$, and its structure can be analyzed using the same tools developed for the Burnside functor. In particular, we show that its composition factors are parametrized by B-groups of p'-order, and we describe the corresponding simple functors explicitly. We introduce this functor as a Grothendieck group as follows.

Let R be a commutative ring with identity and let $\mathbb{K}$ be a field of characteristic zero. Let G be a finite group. A left G-set is said to be *left p-free*, if it is a disjoint union of transitive G-sets with p'-stabilizers. Let ${}_G\mathsf{set}^{\Delta,p}$ denote the category of left p-free G-sets. One can similarly define the category $\mathsf{set}_G^{\Delta,p}$ of right p-free G-sets. Identifying a G-set X with a $(G,1)$-biset induces an isomorphism ${}_G\mathsf{set}^{\Delta,p} \cong {}_G\mathsf{set}_1^{\Delta,p}$.

Let $B^{\Delta,p}(G)$ denote the Grothendieck group of the category ${}_G\mathsf{set}^{\Delta,p}$ with respect to disjoint unions. Note that

$$B^{\Delta,p}(G) = \bigoplus_{H \in [s_G]_{p'}} \mathbb{Z}[G/H]\,,$$

where $[s_G]_{p'}$ denotes a set of representatives of the G-conjugacy classes of p'-subgroups of G. The direct product of G-sets induces a commutative ring structure on $B^{\Delta,p}(G)$. In fact, the direct product of two transitive left p-free G-sets is a disjoint union of such transitive bisets; see, for instance, [Bc00, Equation 3.1.2].

5.1 Remark Let H and K be subgroups of G. Then

$$|(G/K)^H| = |\{x \in G/K \mid H^x \subseteq K\}|\,.$$

In particular, if K is a p'-group, i.e., if $[G/K] \in B^{\Delta,p}(G)$ and if H is not a p'-subgroup, then $|(G/K)^H| = 0$. This means that the mark morphism

$$\phi : B(G) \to \prod_{H \in [s_G]} \mathbb{Z}, \quad [X] \mapsto (|X^H|)_{H \in [s_G]}$$

restricts to a map

$$\phi : B^{\Delta,p}(G) \to \prod_{H \in [s_G]_{p'}} \mathbb{Z}, \quad [X] \mapsto (|X^H|)_{H \in [s_G]_{p'}}$$

which we denote again by ϕ. It follows that the primitive idempotents of $\mathbb{K}B^{\Delta,p}(G)$ are precisely the idempotents $e_H^G \in \mathbb{K}B(G)$ where H is a p'-subgroup of G. Setting

$$e_{p'}^G := \sum_{H \in [s_G]_{p'}} e_H^G$$

we have

$$\mathbb{K}B^{\Delta,p}(G) = e_{p'}^G \mathbb{K}B(G) = \mathbb{K}B(G) e_{p'}^G = e_{p'}^G \mathbb{K}B(G) e_{p'}^G\,.$$

5.2 Remark Note that the commutative ring $B^{\Delta,p}(G)$ does not have an identity element unless G is a p'-subgroup.

We consider the functor $RB^{\Delta,p} : R\mathcal{C}^{\Delta,p} \to {}_R\mathsf{Mod}$ defined as

- $G \mapsto RB^{\Delta,p}(G)$.
- $X \in RB^{\Delta,p}(H,G) \mapsto \big(RB(X) : RB^{\Delta,p}(G) \to RB^{\Delta,p}(H), \quad U \mapsto X \cdot_G U\big)$.

Note that $RB^{\Delta,p}$ is isomorphic to the representable functor $RB^{\Delta,p}(-,1)$ at 1. Also, it is a p-bifree subfunctor of RB.

By Theorem 4.7, the poset of the subfunctors of $\mathbb{K}B^{\Delta,p}$ is isomorphic to the poset of closed subsets of

$$\Theta(\mathbb{K}B^{\Delta,p}) = \{H \in [B^{\Delta}\text{-gr}] \,|\, \mathbf{e}_H^{\Delta,p} \subseteq \mathbb{K}B^{\Delta,p}\}\,.$$

This observation allows us to identify the minimal groups contributing to the functor $\mathbb{K}B^{\Delta}$. The next result shows that they are exactly the B-groups of p'-order, in the sense of Boucs theory.

5.3 Lemma *Let H be a $B^{\Delta,p}$-group. Then $\mathbf{e}_H^{\Delta,p} \subseteq \mathbb{K}B^{\Delta,p}$ if and only if H is a p'-group.*

Proof Suppose $\mathbf{e}_H^{\Delta,p} \subseteq \mathbb{K}B^{\Delta,p}$. Then $e_H^H \in \mathbf{e}_H^{\Delta,p}(H) \subseteq \mathbb{K}B^{\Delta,p}(H)$. By Remark 5.1, H is a p'-group. Conversely, if H is a p'-group, then $e_H^H \in \mathbb{K}B^{\Delta,p}(H)$ and hence $\mathbf{e}_H^{\Delta,p} \subseteq \mathbb{K}B^{\Delta,p}$. □

As a consequence, we obtain the following classification of composition factors.

5.4 Corollary *The composition factors of the p-bifree biset functor $\mathbb{K}B^{\Delta,p}$ are the functors $S_{G,\mathbb{K}}^{\Delta,p}$ where G is a B-group of p'-order.*

6 The simple functor $S_{1,\mathbb{K}}^{\Delta,p}$

We now turn to the simple functor $S_{1,\mathbb{K}}^{\Delta,p}$, corresponding to the trivial group and the trivial module. This functor plays a foundational role in the theory, as it appears as a composition factor in both the Burnside and character functors. In this section, we compute the dimension of $S_{1,\mathbb{K}}^{\Delta,p}(G)$ for an arbitrary finite group G, and show that it is governed by the number of conjugacy classes of cyclic p'-subgroups of G.

Let R be a commutative ring with identity and let $\mathbb{K}$ be a field of characteristic zero. Let (G,V) be a pair of a finite group G and a simple $R\mathrm{Out}(G)$-module V. We first describe the simple functor $S_{G,V}^{\Delta,p}$ in more detail.

Let $E_G := RB^{\Delta,p}(G,G)$ denote the endomorphism algebra of G in the category $R\mathcal{C}^{\Delta,p}$. Then $\mathcal{E}^{\Delta,p}(G) = E_G/I_G \cong R\mathrm{Out}(G)$. We consider V as a simple E_G-module and define the functor $L_{G,V}$ by

$$L_{G,V}(H) = RB^{\Delta,p}(H,G) \otimes_{E_G} V\,.$$

Then by [Bc96], $L_{G,V}$ has a unique maximal subfunctor $J_{G,V}$ whose evaluation at a finite group H is given by

$$J_{G,V}(H) = \Big\{ \sum_i x_i \otimes v_i \in L_{G,V}(H) \,|\, \forall y \in RB^{\Delta,p}(G,H) : \sum_i (y \cdot_H x_i)(v_i) = 0 \Big\} .$$

The simple functor $S^{\Delta,p}_{G,V}$ is defined as the quotient $L_{G,V}/J_{G,V}$.

We compute the $\mathbb{K}$-dimension of $S^{\Delta,p}_{1,\mathbb{K}}(G)$ inspired by the proof of [Bc96, Proposition 8]. Note that the functor $L_{1,\mathbb{K}}$ is isomorphic to the functor $\mathbb{K}B^{\Delta,p}$. Moreover for any finite group G, identifying $L_{1,\mathbb{K}}(G)$ with $\mathbb{K}B^{\Delta,p}(G)$ one has

$$J_{1,\mathbb{K}}(G) = \{ X \in \mathbb{K}B^{\Delta,p}(G) \,|\, \forall Y \in \mathbb{K}B^{\Delta,p}(G) : |G \setminus (Y \times X)| = 0 \} .$$

This implies that the dimension of $S^{\Delta,p}_{1,\mathbb{K}}(G)$ is equal to the rank of the bilinear form

$$\langle -, - \rangle : \mathbb{K}B^{\Delta,p}(G) \times \mathbb{K}B^{\Delta,p}(G) \to \mathbb{K}, \quad (X,Y) \mapsto |G \setminus (Y \times X)| .$$

The set of primitive idempotents $\{e^G_H\}_{H\in[s_G]_{p'}}$ form an orthogonal basis with respect to this bilinear form. Moreover, for $H \in [s_G]_{p'}$, one has

$$\begin{aligned} \langle e^G_H, e^G_H \rangle = |G \setminus e^G_H| &= \frac{1}{|N_G(H)|} \sum_{K \leqslant H} |K| \mu(K,H) \\ &= \frac{1}{|N_G(H)|} \sum_{x \in H} \sum_{\langle x \rangle \leqslant K \leqslant H} \mu(K,H) = \frac{\phi_1(H)}{|N_G(H)|} \end{aligned}$$

where $\phi_1(H)$ is the number of elements $x \in H$ such that $\langle x \rangle = H$. This is non-zero if and only if H is cyclic. This proves the following.

6.1 Theorem *Let G be a finite group. The $\mathbb{K}$-dimension of $S^{\Delta,p}_{1,\mathbb{K}}(G)$ is equal to the number of conjugacy classes of cyclic p'-subgroups of G.*

6.2 Remark Let G be a finite group. Then $\delta_p(G) = 1$ if and only if G is a cyclic p'-group. Thus, Theorem 6.1 is a special case of Theorem 4.11.

7 The functor of complex character ring

Let $\mathbb{C}$ be an algebraically closed field of characteristic zero. We denote by $R_\mathbb{C}(G)$ the Grothendieck group of finite dimensional $\mathbb{C}G$-modules with respect to short exact sequences. For a commutative ring R with unity, we set $\mathrm{R}R_\mathbb{C}(G) := \mathrm{R} \otimes_\mathbb{Z} R_\mathbb{C}(G)$.

We identify $\mathbb{C}R_\mathbb{C}(G)$ with the $\mathbb{C}$-vector space of class functions from G to $\mathbb{C}$. If X is a p-bifree (H,G)-biset, then tensoring with $\mathbb{C}X$ over $\mathbb{C}G$ induces a well-defined $\mathbb{C}$-linear map

$$\mathbb{C}R_\mathbb{C}(X) : \mathbb{C}R_\mathbb{C}(G) \to \mathbb{C}R_\mathbb{C}(H).$$

This endows $\mathbb{C}R_\mathbb{C}(-)$ with a structure of p-bifree biset functor over $\mathbb{C}$.

7.1 Remark Given a pair (G, V) where G is a finite group and V is a simple $\mathbb{C}\mathrm{Out}(G)$-module, let $S_{G,V}$ denote the associated simple biset functor over $\mathbb{C}$. By [Bc10, Corollary 7.3.5], there is a *canonical* decomposition of biset functors

$$\mathbb{C}R_{\mathbb{C}} = \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}\,,$$

where (m, ξ) runs through the set of pairs consisting of a positive integer m and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$. For each finite group G, we therefore view $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(G)$ as the corresponding direct summand (hence a subspace) of $\mathbb{C}R_{\mathbb{C}}(G)$ afforded by the above canonical decomposition. Restricting along the inclusion of the p-bifree biset category yields the same equality of p-bifree biset functors. Note that $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ is not necessarily simple as a p-bifree biset functor.

Our aim in this section is to describe the composition factors of $\mathbb{C}R_{\mathbb{C}}(-)$ as a p-bifree biset functor.

For a cyclic group $\mathbb{Z}/m\mathbb{Z}$ and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$, we denote by $\tilde{\xi}$ the class function from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{C}$ obtained by extending ξ by 0, i.e., for $x \in \mathbb{Z}/m\mathbb{Z}$,

$$\tilde{\xi}(x) = \begin{cases} \xi(x), & \text{if } x \in (\mathbb{Z}/m\mathbb{Z})^\times \\ 0, & \text{otherwise}\,. \end{cases}$$

7.2 Proposition *Let G be a finite group. Let also m be a positive integer and ξ a primitive character of $(\mathbb{Z}/m\mathbb{Z})^\times$. Then for any $\chi \in S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(G)$, there exists $\chi' \in \underline{S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}}(G)$ and $\alpha \in I_G$ such that*

$$\chi = \chi' + \alpha \cdot \chi\,.$$

Proof We divide the proof into several steps.

Step 1: By Proposition 1.5(iv) of Chapter 3 of [BHabil], for any $\chi \in \mathbb{C}R_{\mathbb{C}}(G)$, we have

$$\chi = \frac{1}{|G|} \sum_{\substack{L \leqslant K \leqslant G \\ K \text{ cyclic}}} |L|\mu(L, K)\mathrm{Ind}_L^G \mathrm{Res}_L^G \chi\,.$$

It follows that if G is not a cyclic group, the claim holds by putting $\chi' = 0$ and setting

$$\alpha = \frac{1}{|G|} \sum_{\substack{L \leqslant K \leqslant G \\ K \text{ cyclic}}} |L|\mu(L, K)\Big[\frac{G \times G}{\Delta(L)}\Big] \in I_G.$$

Note that this also shows that $\underline{S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}}(G) = \{0\}$ unless G is cyclic.

Step 2: Now suppose $G = \mathbb{Z}/n\mathbb{Z}$ is a cyclic group. If n is not a multiple of m, then by [Bc10, Corollary 7.4.3] $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(\mathbb{Z}/n\mathbb{Z}) = \{0\}$ and hence there is nothing to prove.

Therefore, assume that n is a multiple of m and let $\chi \in S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(\mathbb{Z}/n\mathbb{Z})$. Then we can write

$$\chi = \tilde{e}_G^G \chi + (1 - \tilde{e}_G^G)\chi.$$

Since $1 - \tilde{e}_G^G \in I_G$, it is sufficient to consider $\tilde{e}_G^G \chi$. But the functor $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ is generated by $\tilde{\xi}$, and suppose that

$$\tilde{e}_G^G \cdot_G \left[\frac{G \times \mathbb{Z}/m\mathbb{Z}}{L}\right] \cdot_{\mathbb{Z}/m\mathbb{Z}} \tilde{\xi}$$

is non-zero for some $L \leqslant G \times \mathbb{Z}/m\mathbb{Z}$. Since $\mathbb{Z}/m\mathbb{Z}$ is a minimal group for the functor $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$, it follows that $p_2(L) = \mathbb{Z}/m\mathbb{Z}$ and $k_2(L) = \{1\}$. Moreover, by [Bc10, Corollary 2.5.12 and Theorem 5.2.4], we have $p_1(L) = G$. It follows that the space $\tilde{e}_G^G S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(G)$ is one dimensional generated by the inflation $\tilde{e}_G^G \mathrm{Inf}_{\mathbb{Z}/m\mathbb{Z}}^G \tilde{\xi}$. Hence we may put $\tilde{e}_G^G \chi = \tilde{e}_G^G \mathrm{Inf}_{\mathbb{Z}/m\mathbb{Z}}^G \tilde{\xi}$.

Let $N \leqslant G$ be such that $G/N \cong \mathbb{Z}/m\mathbb{Z}$. Write $N = N_p \times N_{p'}$. Assume first that $N_{p'} \neq 1$. Then, by [Bc10, Corollary 2.5.12 and Theorem 5.2.4], one has

$$\begin{aligned}
(\tilde{e}_G^G \mathrm{Inf}_{G/N_{p'}}^G \mathrm{Def}_{G/N_{p'}}^G)\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi} &= \tilde{e}_G^G \mathrm{Inf}_{G/N_{p'}}^G (\mathrm{Def}_{G/N_{p'}}^G \tilde{e}_G^G \mathrm{Inf}_{G/N_{p'}}^G) \mathrm{Inf}_{G/N}^{G/N_{p'}} \tilde{\xi} \\
&= m_{G,N_{p'}} \tilde{e}_G^G \mathrm{Inf}_{G/N_{p'}}^G \tilde{e}_{G/N_{p'}}^{G/N_{p'}} \mathrm{Inf}_{G/N}^{G/N_{p'}} \tilde{\xi} \\
&= m_{G,N_{p'}} \tilde{e}_G^G (\widetilde{\mathrm{Inf}_{G/N_{p'}}^G e_{G/N_{p'}}^{G/N_{p'}}}) \mathrm{Inf}_{G/N_{p'}}^G \mathrm{Inf}_{G/N}^{G/N_{p'}} \tilde{\xi} \\
&= m_{G,N_{p'}} \tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi}\,.
\end{aligned}$$

Since G is cyclic, $m_{G,N_{p'}} \neq 0$ by [Bc10, Proposition 5.6.1]. Thus, the condition is satisfied by putting $\chi' = 0$ and $\alpha = \frac{1}{m_{G,N}} \tilde{e}_G^G \mathrm{Inf}_{G/N_{p'}}^G \mathrm{Def}_{G/N_{p'}}^G$. This also shows that $\underline{S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}}(G) = 0$.

Step 3: Now suppose that $N_{p'} = 1$, so N is a cyclic p-group. Note that the restriction of $\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi}$ to a proper subgroup is equal to zero. Hence if G is a p-group, then $\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi} \in \underline{S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}}(G)$. Again, the condition is satisfied in this case too.

Suppose G is not a p-group, then

$$\begin{aligned}
\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi} &= \frac{1}{|G|} \sum_{K \leqslant G} |K| \mu(K,G) \mathrm{Ind}_K^G \mathrm{Res}_K^G \mathrm{Inf}_{G/N}^G \tilde{\xi} \\
&= \frac{1}{|G|} \sum_{K \leqslant G} |K| \mu(K,G) \mathrm{Ind}_K^G \mathrm{Inf}_{K/(K\cap N)}^K \mathrm{Iso}(f^{-1}) \mathrm{Res}_{KN/N}^{G/N} \tilde{\xi} \\
&= \frac{1}{|G|} \sum_{K \leqslant G : KN = G} |K| \mu(K,G) \mathrm{Ind}_K^G \mathrm{Inf}_{K/(K\cap N)}^K \mathrm{Iso}(f^{-1})\, \tilde{\xi}\,,
\end{aligned}$$

where $f : K/K \cap N \to KN/N$ is the canonical isomorphism. Note that since N is a p-subgroup of G, the condition $KN = G$ holds if and only if $G_{p'} \subseteq K$ and $K_pN = G_p$. There are two cases:

(i) If $N < G_p$, then $NK_p = G_p$ holds if and only if $K_p = G_p$, because G_p is a cyclic p-group, and in this case we get $K = G$.

(ii) If $N = G_p$, then K_p can be any subgroup of G_p, so K is any subgroup with $G_{p'} \subseteq K$.

In Case (i), we get

$$\begin{aligned}\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi} &= \frac{1}{|G|} \sum_{K \leqslant G : KN = G} |K| \mu(K,G) \mathrm{Ind}_K^G \mathrm{Inf}_{K/(K\cap N)}^K \mathrm{Iso}(f^{-1})\, \tilde{\xi} \\ &= \mathrm{Inf}_{G/N}^G \tilde{\xi}\,.\end{aligned}$$

On the other hand, in Case (ii), we obtain

$$\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi} = \frac{1}{|G|} \sum_{G_{p'} \leqslant K \leqslant G} |K| \mu(K,G) \mathrm{Ind}_K^G \mathrm{Inf}_{K/(K\cap N)}^K \mathrm{Iso}(f^{-1})\, \tilde{\xi}\,.$$

Here we have $\mu(K,G) = \mu(|G:K|)$, the number theoretic Möbius function. We have $\mu(|G:K|) = 1$ if $G = K$, it is -1 if $|G:K| = p$ and 0 otherwise. Hence the above equality becomes

$$\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi} = \mathrm{Inf}_{G/N}^G \mathrm{Iso}\, \tilde{\xi} - \frac{1}{p} \mathrm{Ind} \mathrm{Inf}_{K/(K\cap N)}^G \mathrm{Iso}(f^{-1})\, \tilde{\xi}\,,$$

where K is the unique subgroup of index p of G. Now for any non-trivial p'-subgroup M of G, we have

$$\mathrm{Def}_{G/M}^G \mathrm{Inf}_{G/N}^G \tilde{\xi} = \mathrm{Inf}_{G/NM}^{G/M} \mathrm{Def}_{G/NM}^{G/N} \tilde{\xi} = 0$$

since $|G:NM| < |G:N| = m$ and since $G/N \cong \mathbb{Z}/m\mathbb{Z}$ is a minimal group for the functor $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$. Hence $\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi} \in \underline{S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}}(G)$ in Case (i).

For Case (ii), we need to evaluate

$$\mathrm{Def}_{G/M}^G \Big(\mathrm{Inf}_{G/N}^G \mathrm{Iso}\, \xi - \frac{1}{p} \mathrm{Ind} \mathrm{Inf}_{K/(K\cap N)}^G \mathrm{Iso}(f^{-1}) \xi \Big)\,.$$

The first term is zero, by the above calculations. We also evaluate the second term:

$$\mathrm{Def}_{G/M}^G \mathrm{Ind} \mathrm{Inf}_{K/(K\cap N)}^G \mathrm{Iso}(f^{-1})\, \tilde{\xi} = \mathrm{Ind} \mathrm{Inf}_{K/(M(K\cap N))}^{G/M} \mathrm{Def}_{K/(M(K\cap N))}^{K/(K\cap N)} \mathrm{Iso}(f^{-1})\, \tilde{\xi} = 0$$

since $|K : M(K \cap N)| < |K : K \cap N| = m$. Hence $\tilde{e}_G^G \mathrm{Inf}_{G/N}^G \tilde{\xi} \in \underline{S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}}(G)$ in Case (ii) too. The result follows. □

7.3 Notation Let $m \in \mathbb{N}$ and let ξ be a primitive character of $(\mathbb{Z}/m\mathbb{Z})^\times$. Let $n \in \mathbb{N}_0$ and let $G = \mathbb{Z}/mp^n\mathbb{Z}$ be the cyclic group of order $m \cdot p^n$. We set $\xi_{m,n,p} := \mathrm{Inf}_{(\mathbb{Z}/m\mathbb{Z})^\times}^{G^\times} \xi$ and $\tilde{\xi}_{m,n,p} := \tilde{e}_G^G \mathrm{Inf}_{\mathbb{Z}/m\mathbb{Z}}^G \tilde{\xi}$.

Note that for $x \in G$, one has

$$\tilde{\xi}_{m,n,p}(x) = \begin{cases} \mathrm{Inf}_{\mathbb{Z}/m\mathbb{Z}}^G \tilde{\xi}(x), & \text{if } \langle x \rangle = G\,, \\ 0, & \text{otherwise.} \end{cases}$$

7.4 Definition Let $k \in \mathbb{N}$. We call an irreducible character χ of $(\mathbb{Z}/k\mathbb{Z})^\times$ *p-primitive*, if χ is inflated from a primitive character of $(\mathbb{Z}/m\mathbb{Z})^\times$ for some $m \mid k$ such that k/m is a power of p, that is, $\chi = \mathrm{Inf}_{(\mathbb{Z}/m\mathbb{Z})^\times}^{(\mathbb{Z}/k\mathbb{Z})^\times} \xi$ for some primitive character ξ.

Note that the character $\xi_{m,n,p}$ above is p-primitive.

7.5 Corollary *Let G be a finite group, m a positive integer, and ξ a primitive character of $(\mathbb{Z}/m\mathbb{Z})^\times$.*

(a) $\underline{S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}}(G) = \{0\}$ *unless G is a cyclic group of order mp^n for some integer $n \geqslant 0$.*

(b) *If G is a cyclic group of order mp^n for some $n \geqslant 0$, then $\underline{S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}}(G)$ is one-dimensional generated by $\tilde{\xi}_{m,n,p}$.*

Proof Part (a) follows from Steps 1 and 2 in the proof of Proposition 7.2, while Part (b) follows from Step 3. ∎

Now we show that the functor $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ is semisimple as a p-bifree biset functor.

7.6 Theorem *Let m be a positive integer and let ξ be a primitive character of $(\mathbb{Z}/m\mathbb{Z})^\times$. Then*

$$S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi} \cong \bigoplus_{n \in \mathbb{N}_0} S^{\Delta,p}_{\mathbb{Z}/mp^n\mathbb{Z},\mathbb{C}_{\xi_{m,n,p}}}$$

as p-bifree biset functors.

Proof As before we regard $S := S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ as a subfunctor of $\mathbb{C}R_\mathbb{C}$. By Proposition 7.2, Corollary 7.5 and Theorem 10.2, the composition factors of S are exactly the functors $S^{\Delta,p}_{\mathbb{Z}/mp^n\mathbb{Z},\mathbb{C}_{\xi_{m,n,p}}}$ each with multiplicity one. We will show that each of these functors appears as a subfunctor. Let F be a p-bifree biset subfunctor of S and let G be a minimal group for F. Since $F(K) = 0$ for any group K with $|K| < |G|$, we have $0 \neq F(G) \subseteq \underline{S}(G)$. By Corollary 7.5, it follows that G is a cyclic group of order $m \cdot p^n$ for some natural number n, and that $F(G)$ is one-dimensional generated by $\tilde{\xi}_{m,n,p} = \tilde{e}_G^G \mathrm{Inf}_{\mathbb{Z}/m\mathbb{Z}}^G \xi$. Conversely, if G is a cyclic group of order $m \cdot p^n$, then $\underline{S}(G) \neq 0$ and the subfunctor generated by $\underline{S}(G)$ has a minimal group G.

For any natural number $n \geqslant 0$, let $G_n = \mathbb{Z}/mp^n\mathbb{Z}$ be a cyclic group of order $m \cdot p^n$ and let F_n be the subfunctor of S generated by $\tilde{\xi}_{m,n,p}$. Then by the discussions above F_n has a minimal group G_n. We claim that F_n is a simple p-bifree biset functor. It is sufficient to show that $F_{n+k} \not\subseteq F_n$ for any $k \geqslant 1$. Indeed, if F_n has a non-zero subfunctor M, then the minimal group of M is G_{n+k} for some k and by definition $F_{n+k} \subseteq M$.

Let $k \geqslant 1$. Since $|G_{n+k}|/|G_n|$ is a p-power, the space $\mathbb{C}B^{\Delta,p}(G_{n+k}, G_n) \circ F_n(G_n)$ is one-dimensional, generated by $\mathrm{Ind}_{G_n}^{G_{n+k}} \tilde{\xi}_{m,n,p}$. In particular, $\tilde{e}_{G_{n+k}}^{G_{n+k}} F_n(G_{n+k}) = 0$. But, $\tilde{e}_{G_{n+k}}^{G_{n+k}} F_{n+k}(G_{n+k}) = F_{n+k}(G_{n+k}) \neq 0$. Hence, $F_{n+k} \not\subseteq F_n$, as required. This shows that F_n is simple. Moreover, for any $x \in (\mathbb{Z}/mp^n)^\times$, we have $x \cdot \tilde{\xi}_{m,n,p} = \xi_{m,n,p}(x)\tilde{\xi}_{m,n,p}$ which implies $F_n(G_n) \cong \mathbb{C}_{\xi_{m,n,p}}$. This shows that $F_n \cong S^{\Delta,p}_{G_n,\mathbb{C}_{\xi_{m,n,p}}}$ and theorem is proved. □

These calculations show that $\mathbb{C}R_\mathbb{C}$ decomposes into a direct sum of simple p-bifree biset functors indexed by pairs (m, ξ), where m is a positive integer and ξ is a p-primitive character of $(\mathbb{Z}/m\mathbb{Z})^\times$.

7.7 Corollary *One has*

$$\mathbb{C}R_\mathbb{C} \cong \bigoplus_{(m,\xi)} S^{\Delta,p}_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$$

where (m, ξ) runs through the set of pairs consisting of a positive integer m and a p-primitive character ξ of $(\mathbb{Z}/m\mathbb{Z})^\times$.

Proof By Remark 7.1 and Theorem 7.6, we have

$$\mathbb{C}R_\mathbb{C} \cong \bigoplus_{(k,\chi)} \bigoplus_{n\in\mathbb{N}_0} S^{\Delta,p}_{\mathbb{Z}/kp^n\mathbb{Z},\mathbb{C}_{\chi_{k,n,p}}}$$

where (k, χ) runs through the set of pairs consisting of a positive integer k and a primitive character χ of $(\mathbb{Z}/k\mathbb{Z})^\times$. We may write the above sum as

$$\mathbb{C}R_\mathbb{C} \cong \bigoplus_{m\in\mathbb{N}} \bigoplus_{(l,\chi)} S^{\Delta,p}_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_{\chi_{m/p^l,l,p}}}$$

where (l, χ) runs through the set of pairs of a non-negative integer l with $p^l \mid m$ and a primitive character χ of $(\mathbb{Z}/(m/p^l)\mathbb{Z})^\times$. By the definition of $\chi_{m,n,p}$, it follows that as p^l runs over p-power divisors of m, the characters $\chi_{m/p^l,l,p}$ runs over all p-primitive characters of $(\mathbb{Z}/m\mathbb{Z})^\times$. The result follows. □

We now present an alternative proof of Theorem 7.6 and Corollary 7.7. This proof was suggested by an anonymous referee, whom we thank for this valuable contribution.

7.8 Theorem *The p-bifree biset functor $\mathbb{C}R_\mathbb{C}$ is semisimple.*

Proof (Referee) Let G and H be finite groups and let U be a p-bifree (H,G)-biset. For a class map $\varphi : G \to \mathbb{C}$, we denote by $U\varphi$ the map $\mathbb{C}R_{\mathbb{C}}(U)(\varphi) : H \to \mathbb{C}$. By [Bc10, Lemma 7.1.3], this map is given by

$$\forall h \in H, \quad (U\varphi)(h) = \frac{1}{|G|} \sum_{\substack{u \in U \\ g \in G \\ hu = ug}} \varphi(g)\,.$$

Denote by $\langle -, - \rangle_G$ the standard inner prouct on $\mathbb{C}R_{\mathbb{C}}(G)$ and note that for any $\theta \in \mathbb{C}R_{\mathbb{C}}(H)$, one has

$$\langle U\varphi, \theta \rangle_H = \langle \varphi, U^{\mathrm{op}}\theta \rangle_G\,,$$

where U^{op} denotes the opposite biset. It follows that if F is a subfunctor of $\mathbb{C}R_{\mathbb{C}}$, then the map

$$G \mapsto F^{\perp}(G) := \{\psi \in \mathbb{C}R_{\mathbb{C}}(G) \,|\, \forall \varphi \in F(G),\, \langle \psi, \varphi \rangle_G = 0\}$$

defines a subfunctor $F^{\perp}$ of $\mathbb{C}R_{\mathbb{C}}$. Moreover, for any finite group G, since the product $\langle -, - \rangle_G$ is positive, we have $F(G) \cap F^{\perp}(G) = \{0\}$. It follows that $F(G) \oplus F^{\perp}(G) = \mathbb{C}R_{\mathbb{C}}(G)$, since the product $\langle -, - \rangle_G$ is non-degenerate. In other words $F \oplus F^{\perp} = \mathbb{C}R_{\mathbb{C}}$.

It follows more generally that if $F' \leqslant F$ are subfunctors of $\mathbb{C}R_{\mathbb{C}}$, then $F' \leqslant F \leqslant F \oplus F'$ so $F = F' \oplus (F \cap F'^{\perp})$, and $F \cap F'^{\perp}$ is a subfunctor of F isomorphic to F/F'. So any subquotient functor of $\mathbb{C}R_{\mathbb{C}}$ is in fact isomorphic to a subfunctor of $\mathbb{C}R_{\mathbb{C}}$.

Now let Σ denote the sum of all simple subfunctors of $\mathbb{C}R_{\mathbb{C}}$. Then $\Sigma \oplus \Sigma^{\perp} = \mathbb{C}R_{\mathbb{C}}$. Suppose that $\Sigma^{\perp} \neq \{0\}$ and let $T = F/F'$ be a composition factor of $\Sigma^{\perp}$, where $F' < F$ are subfunctors of $\Sigma^{\perp}$. Then there is a simple subfunctor T_1 of F, isomorphic to T, such that $F = F' \oplus T_1$. But then T_1 is a simple subfunctor of $\mathbb{C}R_{\mathbb{C}}$, so $T_1 \leqslant \Sigma$. Thus

$$\{0\} \neq T_1 \leqslant \Sigma \cap F \leqslant \Sigma \cap \Sigma^{\perp} = \{0\}\,.$$

This contradiction shows that $\Sigma = \mathbb{C}R_{\mathbb{C}}$, i.e., $\mathbb{C}R_{\mathbb{C}}$ is semisimple. □

An alternative proof of Theorem 7.6 (Referee): Let m be a positive integer and let $\xi : (\mathbb{Z}/m\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ be a primitive character. Then the simple biset functor $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_{\xi}}$ is a direct summand of $\mathbb{C}R_{\mathbb{C}}$, and by Theorem 7.8, $\mathbb{C}R_{\mathbb{C}}$ is semisimple as a p-bifree biset functor. It follows that $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_{\xi}}$ is also semisimple p-bifree biset functor.

Let $S^{\Delta}_{H,V}$ be a simple direct summand of $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_{\xi}}$. Let f be a non-zero element of $S^{\Delta}_{H,V}(H)$ and $x \in H$ such that $f(x) \neq 0$. Since H is a minimal group for $S^{\Delta}_{H,V}$, it follows that $\langle x \rangle = H$, so H is cyclic. Set $n = |H|$. Now V is a simple module for $\mathrm{Out}(H) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$, so $V = \mathbb{C}_{\lambda}$ for some character $\lambda : (\mathbb{Z}/m\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$. The action of $\alpha \in (\mathbb{Z}/n\mathbb{Z})^{\times}$ on $V = \mathbb{C}_{\lambda}$ is given by $\alpha \cdot 1_{\mathbb{C}_{\lambda}} = \lambda(\alpha) 1_{\mathbb{C}_{\lambda}}$. It follows that $\alpha \cdot f = \lambda(\alpha) f$, and we can assume that $f = \tilde{\lambda}$, where $\tilde{\lambda} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ is equal to λ on $(\mathbb{Z}/n\mathbb{Z})^{\times}$ and to 0 on $\mathbb{Z}/n\mathbb{Z} \setminus (\mathbb{Z}/n\mathbb{Z})^{\times}$.

But $\tilde{\lambda} \in S^{\Delta,p}_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\lambda}(\mathbb{Z}/n\mathbb{Z}) \subseteq S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(\mathbb{Z}/n\mathbb{Z})$. So, in particular, $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(\mathbb{Z}/n\mathbb{Z}) \neq 0$ and hence $m \,|\, n$ by [Bc10, Corollary 7.4.3]. Moreover, since $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ is simple, the biset subfunctor $\langle\tilde{\lambda}\rangle$ of $\mathbb{C}R_{\mathbb{C}}$ generated by $\tilde{\lambda}$ is equal to $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$. This in turn is generated by the map $\tilde{\xi} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$. In particular,

$$\mathbb{C}B(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}/n\mathbb{Z})(\tilde{\lambda}) = \langle\tilde{\lambda}\rangle(\mathbb{Z}/m\mathbb{Z}) = S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(\mathbb{Z}/m\mathbb{Z}) = \mathbb{C}\tilde{\xi}\,.$$

Equivalently, there exist a subgroup $L \leqslant \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and a non-zero scalar $a \in \mathbb{C}$ such that $a\tilde{\xi} = \left((\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})\,/L\right)(\tilde{\lambda})$. Setting $A := p_1(L)$, $B := k_1(L)$, $C := p_2(L)$ and $D := k_2(L)$, we have

$$a\tilde{\xi} = \mathrm{Ind}_A^{\mathbb{Z}/m\mathbb{Z}}\mathrm{Inf}_{A/B}^A\mathrm{Iso}(\varphi)\mathrm{Def}_{C/D}^C\mathrm{Res}_C^{\mathbb{Z}/n\mathbb{Z}}(\tilde{\lambda})\,,$$

where $\varphi : C/D \to A/B$ is the canonical group isomorphism. Since this is non-zero and $\mathbb{Z}/m\mathbb{Z}$ is a minimal group for $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$, we have $A = \mathbb{Z}/m\mathbb{Z}$ and $B = 1$. Moreover, since $\mathbb{Z}/n\mathbb{Z}$ is a minimal group for $S^{\Delta,p}_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\lambda}$, any proper restriction of $\tilde{\lambda}$ is equal to 0. It follows that $C = \mathbb{Z}/n\mathbb{Z}$ and $C/D \cong A/B \cong \mathbb{Z}/m\mathbb{Z}$, so D is the unique subgroup of $\mathbb{Z}/n\mathbb{Z}$ of index n/m. Finally,

$$a\tilde{\xi} = \mathrm{Iso}(\varphi)\mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}}(\tilde{\lambda})\,,$$

where φ is an automorphism of $\mathbb{Z}/m\mathbb{Z}$. Equivalently, there exists a non-zero scalar c such that

$$c\tilde{\xi} = \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}}(\tilde{\lambda})\,. \tag{2}$$

However, again, since $\mathbb{Z}/n\mathbb{Z}$ is a minimal group for $S^{\Delta,p}_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\lambda}$, any proper deflation of $\tilde{\lambda}$ with p'-kernel is equal to zero. It follows that n/m is a power of p. So, $n = mp^d$ for some $d \in \mathbb{N}_0$.

Now let $\pi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the projection map and let $\alpha \in \mathrm{Aut}(\mathbb{Z}/m\mathbb{Z})$ and $\beta \in \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ such that $\alpha \circ \pi = \pi \circ \beta$. Identifying $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ and $\mathrm{Aut}(\mathbb{Z}/m\mathbb{Z})$ with $(\mathbb{Z}/n\mathbb{Z})^\times$ and $(\mathbb{Z}/m\mathbb{Z})^\times$, respectively, we get that

$$\begin{aligned}
c\xi(\alpha)\tilde{\xi} &= \mathrm{Iso}(\alpha)(c\tilde{\xi})\\
&= \mathrm{Iso}(\alpha) \circ \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}}(\tilde{\lambda})\\
&= \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \circ \mathrm{Iso}(\beta)(\tilde{\lambda})\\
&= \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}}\left(\lambda(\beta)\tilde{\lambda}\right)\\
&= \lambda(\beta)c\tilde{\xi}\,,
\end{aligned}$$

where the second equality follows from Equation (2). It follows that $\xi(\alpha) = \lambda(\beta)$. In particular, $\lambda(\beta) = 1$, if α is the identity, i.e., if β is in the kernel of the projection map $(\pi^\times)^n_m : (\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times$. Moreover, $\lambda = \xi \circ (\pi^\times)^n_m$.

We finally get that any simple p-bifree biset subfunctor of $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ is isomorphic to $S^{\Delta,p}_{\mathbb{Z}/mp^d\mathbb{Z},\mathbb{C}_{\xi_{m,d,p}}}$, for some $d \in \mathbb{N}_0$, where $\xi_{m,d,p} = \xi \circ (\pi^\times)_n^{mp^d}$.

Conversely, given m and d, set $\xi_{m,d,p} = \xi \circ (\pi^\times)_n^{mp^d}$. Then we have

$$\tilde{\xi}_{m,d,p} = \tilde{e}^{\mathbb{Z}/mp^d\mathbb{Z}}_{\mathbb{Z}/mp^d\mathbb{Z}} \mathrm{Inf}^{\mathbb{Z}/mp^d\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \tilde{\xi}\,,$$

so $\tilde{\xi}_{m,d,p} \in S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(\mathbb{Z}/mp^d\mathbb{Z})$. Since the p-bifree biset functor generated by $\tilde{\xi}_{m,d,p}$ is equal to $S^{\Delta,p}_{\mathbb{Z}/mp^d\mathbb{Z},\mathbb{C}_{\xi_{m,d,p}}}$, it follows that $S^{\Delta,p}_{\mathbb{Z}/mp^d\mathbb{Z},\mathbb{C}_{\xi_{m,d,p}}}$ is a subfunctor of $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ for any $d \in \mathbb{N}_0$. Since also $\tilde{\xi}_{m,d,p}$ is uniquely defined by m, ξ, and d, there exists a unique subfunctor of $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ isomorphic to $S^{\Delta,p}_{\mathbb{Z}/mp^d\mathbb{Z},\mathbb{C}_{\xi_{m,d,p}}}$, for each $d \in \mathbb{N}_0$. This proves Theorem 7.6. □

8 The functor of Brauer characters

Let k be an algebraically closed field of characteristic p. We denote by $R_k(G)$ the Grothendieck group of finite dimensional kG-modules with respect to short exact sequences. For a commutative ring R with unity, we set $\mathrm{R}R_k(G) := \mathrm{R} \otimes_{\mathbb{Z}} R_k(G)$.

Let $\mathbb{C}$ be an algebraically closed field of characteristic zero. We identify $\mathbb{C}R_k(G)$ with the $\mathbb{C}$-vector space of class functions from $G_{p'}$ to $\mathbb{C}$. If H is another finite group and X is a p-bifree (H, G)-biset, then kX is projective, and therefore flat, as a right kG-module. Consequently, tensoring with kX over kG induces a well-defined group homomorphism

$$R_k(X) := kX \otimes_{kG} - : R_k(G) \to R_k(H),$$

and a $\mathbb{C}$-linear map

$$\mathbb{C}R_k(X) : \mathbb{C}R_k(G) \to \mathbb{C}R_k(H).$$

This endows $\mathbb{C}R_k(-)$ with a structure of p-bifree biset functor over $\mathbb{C}$.

8.1 Remark Note that the restrictions of the functors $\mathbb{C}R_k$ and $\mathbb{C}R_{\mathbb{C}}$ to the full subcategory of p'-groups are equal via the identifications above. Also, for a positive p'-integer m and a primitive character $\xi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}^\times$, the restriction of the simple functor $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ is simple. Therefore, one has equalities

$$\mathbb{C}R_k = \mathbb{C}R_{\mathbb{C}} = \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$$

of p-bifree biset functors on the full subcategory of p'-groups, where (m, ξ) runs through the set of pairs consisting of a positive p'-integer m and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$.

In this section, we examine the Brauer character ring $\mathbb{C}R_k$ as a p-bifree biset functor. We classify its composition factors explicitly and describe its relationship with the complex character ring functor $\mathbb{C}R_{\mathbb{C}}$. The main result is a short exact sequence of p-bifree biset functors, which reveals how the modular and ordinary character theories fit together within this categorical framework.

8.2 Remark Recall that one has isomorphisms $\mathbb{C}R_k(-) \cong \mathbb{C}\mathrm{Proj}(-) \cong S_{1,1,\mathbb{C}}$ of simple diagonal p-permutation functors over $\mathbb{C}$, see [BY20, Theorem 5.18] and [BY25, Section 6].

We start by showing that the functor $\mathbb{C}R_k$ is semisimple. The proof presented is also suggested by the anonymous referee, simplifying our previous more involved proof.

8.3 Theorem *$\mathbb{C}R_k$ is a semisimple p-bifree biset functor.*

Proof (Referee) Let G be a finite group. Let $\mathrm{res}_G : \mathbb{C}R_{\mathbb{C}}(G) \to \mathbb{C}R_k(G)$ denote the restriction map sending $f \in \mathbb{C}R_{\mathbb{C}}(G)$ to its restriction to $G_{p'}$. Let $\mathrm{pro}_G : \mathbb{C}R_k(G) \to \mathbb{C}R_{\mathbb{C}}(G)$ denote the extension map, sending $f \in \mathbb{C}R_k(G)$ to the map equal to f on $G_{p'}$ and to 0 on $G \setminus G_{p'}$. Then the composition $\mathrm{res}_G \circ \mathrm{pro}_G$ is the identity map on $\mathbb{C}R_k(G)$.

Let H also be a finite group and U a p-bifree (H, G)-biset. Note that if $h \in H$, $g \in G$ and $u \in U$ are such that $hu = ug$, then $h \in H_{p'}$ if and only if $g \in G_{p'}$. Indeed, if n is the order of h, then $h^n u = u = ug^n$. So, g^n is in the right stabilizer of u which is a p'-group. If $h \in H_{p'}$, then n is coprime to p and so $g \in G_{p'}$. A similar argument shows that if $g \in G_{p'}$, then $h \in H_{p'}$. It follows from this observation and from [Bc10, Lemma 7.1.3] that if $f \in \mathbb{C}R_k(G)$, then $Uf \in \mathbb{C}R_{\mathbb{C}}(H)$. Moreover, $\mathrm{pro}_{\mathrm{H}}(Uf) = U\mathrm{pro}_G(f)$. This implies that the maps pro_G form a morphism of p-bifree biset functors $\mathrm{pro} : \mathbb{C}R_k \to \mathbb{C}R_{\mathbb{C}}$. Similarly, the maps res_G form a morphism of p-bifree biset functors $\mathrm{res} : \mathbb{C}R_{\mathbb{C}} \to \mathbb{C}R_k$. The composition $\mathrm{res} \circ \mathrm{pro}$ is equal to the identity of $\mathbb{C}R_k$. The result follows since the functor $\mathbb{C}R_{\mathbb{C}}$ is semisimple by Theorem 7.8. ∎

8.4 Corollary *One has*

$$\mathbb{C}R_k \cong \bigoplus_{(m,\xi)} S^{\Delta,p}_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$$

where (m, ξ) runs through a set of pairs consisting of a positive p'-number m and a primitive character ξ of $(\mathbb{Z}/m\mathbb{Z})^\times$.

Proof (Referee) By Corollary 7.7 and by the proof of Theorem 8.3, composition factors of $\mathbb{C}R_k$ are of the form $S^{\Delta,p}_{\mathbb{Z}/m\mathbb{Z},\xi_{m,n,p}}$ where m is a positive integer and ξ is a primitive character of $(\mathbb{Z}/m\mathbb{Z})^\times$. Such a functor is a direct summand of $\mathbb{C}R_k$ if and only if the map $\tilde{\xi}_{m,n,p}$ vanishes on p-singular elements, i.e., if and only if m is coprime to p and $n = 0$. The result follows. ∎

We now present an alternative proof of Corollary 8.4. We start with the description of the restriction kernels.

8.5 Proposition *Let G be a finite group.*

(i) *One has $\underline{\mathbb{C}R_k}(G) = 0$ unless G is a cyclic p'-group.*

(ii) *Let $G = \mathbb{Z}/m\mathbb{Z}$ be a cyclic p'-group. Then the $\mathbb{C}\mathrm{Aut}(G)$-module $\underline{\mathbb{C}R_k}(G)$ is equal to the direct sum of all primitive $\mathbb{C}\mathrm{Aut}(G)$-modules.*

Proof (i) Let $\chi \in \mathbb{C}R_k(G)$ be a non-zero element and let $g \in G$ with $\chi(g) \neq 0$. Then g is a p'-element and $\mathrm{Res}^G_{\langle g\rangle}\chi \neq 0$. This shows that $\chi \notin \underline{\mathbb{C}R_k}(G)$ and therefore, $\underline{\mathbb{C}R_k}(G) = 0$ if G is not a cyclic p'-group.

(ii) Let $G = \mathbb{Z}/m\mathbb{Z}$ be a cyclic p'-group. Since G is a p'-group, we may identify $\mathbb{C}R_k(G)$ with $\mathbb{C}R_{\mathbb{C}}(G)$, and by Lemma 10.1, $\underline{\mathbb{C}R_k}(G)$ with $\underline{\mathbb{C}R_{\mathbb{C}}}(G)$. By Remark 7.1, one has

$$\mathbb{C}R_k(G) = \left(\bigoplus_{\xi} S_{G,\mathbb{C}_\xi}(G)\right) \oplus \left(\bigoplus_{(H,\eta):|H|<|G|} S_{H,\mathbb{C}_\eta}(G)\right)$$

where ξ runs over primitive characters $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ and where (H,η) runs over a pair of cyclic p'-group $H = \mathbb{Z}/n\mathbb{Z}$ with $|H| < |G|$ and a primitive character $\eta : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$. It is easy to see that the space $\oplus_\xi S_{G,\mathbb{C}_\xi}(G) \cong \oplus_\xi \mathbb{C}_\xi$ is contained in $\underline{\mathbb{C}R_k}(G)$. Conversely, let $x \in \underline{\mathbb{C}R_k}(G)$ be a non-zero element and assume that $x \notin \oplus_\xi S_{G,\mathbb{C}_\xi}(G)$. Let H be a group of minimal order with the property that for some η, the (H,η)-coordinate $x_{H,\eta} \in S_{H,\mathbb{C}_\eta}(G)$ is not zero. Then $|H| < |G|$, and $x_{H,\eta} \in \underline{\mathbb{C}R_k}(G)$. Since $S_{H,\mathbb{K}_\eta}$ is simple, the subfunctor of $S_{H,\eta}$ generated by $x_{H,\eta}$ must be equal to $S_{H,\eta}$. It follows that $\mathbb{C}_\eta \cong S_{H,\mathbb{C}_\eta}(H)$ is equal to $\mathbb{C}B^{\Delta,p}(H,G) \circ x_{H,\eta}$. In particular, $x_{H,\eta}$ is not in the restriction kernel $\underline{\mathbb{C}R_k}(G)$, a contradiction. This completes the proof. □

An alternative proof of Corollary 8.4: By Theorem 8.3 and Proposition 10.4, for any finite group G and any $\chi \in \mathbb{C}R_k(G)$, there exist $\chi' \in \underline{\mathbb{C}R_k}(G)$ and $\alpha \in I_G$ satisfying the equality

$$\chi = \chi' + \alpha \cdot \chi\,.$$

It follows from Theorem 10.2 that the multiplicity of a simple p-bifree biset functor $S^{\Delta,p}_{H,V}$ in $\mathbb{C}R_k$ as a composition factor is equal to that of the $\mathbb{C}\mathrm{Aut}(H)$-module V in $\underline{\mathbb{C}R_k}(H)$.

By Proposition 8.5, the restriction kernel is zero unless H is a cyclic p'-group. Hence the multiplicity of $S^{\Delta,p}_{H,V}$ in $\mathbb{C}R_k$ is equal to 0 if H is not a cyclic p'-group. Furthermore, if H is a cyclic p'-group, by Proposition 8.5(ii), the multiplicity of $S^{\Delta,p}_{H,V}$ in $\mathbb{C}R_k$ is non-zero if and only if V is primitive, in which case it is equal to 1. □

8.6 Remark By Corollaries 8.4 and 7.7, we obtain a short exact sequence

$$0 \longrightarrow \bigoplus_{(m,\xi)} S^{\Delta,p}_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi} \longrightarrow \mathbb{C}R_{\mathbb{C}} \longrightarrow \mathbb{C}R_k \longrightarrow 0$$

of p-bifree biset functors, where (m,ξ) runs through the set of pairs consisting of a positive integer m divisible by p and a p-primitive character ξ of $(\mathbb{Z}/m\mathbb{Z})^\times$.

9 Simple biset functors associated with B-groups

In this section, we analyze how certain classical simple biset functors behave under restriction to the p-bifree biset category. Our focus is on the simple functors $S_{K,\mathbb{C}}$ where K is a B-group. As special cases we consider the functors $S_{C_p\times C_p,\mathbb{C}}$ and $S_{C_q\times C_q,\mathbb{C}}$, for a prime number q with $q \neq p$, whose evaluations are closely related to the structure of the Dade group (see [Bc10, Chapters 11–12]). It turns out that both functors decompose as direct sums of simple p-bifree biset functors indexed by specific families of p-groups.

9.1 Remark Let K be a B-group. Let $\mathbf{e}_K$ and $\mathbf{j}_K$ be biset subfunctors of the Burnside ring functor introduced in [Bc10, Section 5], similar to the functors $\mathbf{e}_K^{\Delta,p}$ and $\mathbf{j}_K^{\Delta,p}$ in Section 4. By the proof of [Bc10, Theorem 5.5.4], the functor $S_{K,\mathbb{C}}$ is isomorphic as biset functors to the quotient $\mathbf{e}_K/\mathbf{j}_K$. We identify $S_{K,\mathbb{C}}$ as a subquotient of the Burnside functor via that isomorphism.

The following theorem, suggested by an anonymous referee, simultaneously generalizes the two special cases presented later in this section (see Parts (b) and (c) of Corollary 9.3 below). Our original approach to these cases, consistent with the previous sections, uses restriction kernels. We outline this approach at the end of the section.

9.2 Theorem *(Referee) Let K be a B-group. Then the p-bifree biset functor $S_{K,\mathbb{C}}$ is semisimple. More precisely,*

$$S_{K,\mathbb{C}} \cong \bigoplus_H S_{H,\mathbb{C}}^{\Delta,p}$$

where H runs over a set of isomorphism classes of $B^{\Delta,p}$-groups with the property $\beta(H) \cong K$.

Proof Let $S_{H,V}^{\Delta,p}$ be a simple subquotient of $S_{K,\mathbb{C}}$. Then $S_{H,V}^{\Delta,p}$ is a simple subquotient of $\mathbb{C}B$ as a p-bifree biset functor, so by Corollary 4.9, H is a $B^{\Delta,p}$-group and V is the trivial module $\mathbb{C}$.

Now $S_{K,\mathbb{C}} \cong \mathbf{e}_K/\mathbf{j}_K$, and for a finite group G, the space $\mathbf{e}_K(G)$ has a basis consisting of the idempotents e_X^G, for $X \leqslant G$ such that $X \twoheadrightarrow K$, i.e., $\beta(X) \twoheadrightarrow K$, since K is a B-group. The space $\mathbf{j}_K(G)$ has a basis consisting of the idempotents e_X^G such that $\beta(X) \twoheadrightarrow K$ but $\beta(X) \ncong K$.

Since $S_{H,V}^{\Delta,p}$ is a subquotient of $S_{K,\mathbb{C}}$, we have that $\tilde{e}_H^H S_{H,\mathbb{C}}^{\Delta,p}(H) \cong \mathbb{C}$ is a subquotient of $\tilde{e}_H^H S_{K,\mathbb{C}}(H)$, hence $\tilde{e}_H^H S_{K,\mathbb{C}}(H) \neq 0$. So there exists a subgroup X of H with $\tilde{e}_H^H e_X^H = e_X^H \in \mathbf{e}_K(H) - \mathbf{j}_K(H)$. This implies $\beta(X) \cong K$. Since $\tilde{e}_H^H e_X^H \neq 0$, we have $X = H$, so $\beta(H) \cong K$.

Now $\mathbf{e}_H^{\Delta,p} \leqslant \mathbf{e}_K$, and $\mathbf{e}_H^{\Delta,p} \nleqslant \mathbf{j}_K$. Let F be a subfunctor of $\mathbf{e}_H^{\Delta,p}$, not contained in $\mathbf{j}_K$. Then there exists a group G and a subgroup X of G such that $e_X^G \in F(G) \leqslant \mathbf{e}_H^{\Delta,p}(G)$, but $e_X^G \notin \mathbf{j}_K(G)$. Since $\mathrm{Res}_X^G(e_X^G)$ is a non-zero multiple of e_X^X, and since $\mathrm{Ind}_X^G(e_X^X)$ is a non-zero multiple of e_X^G, we can assume that $X = G$. Since $e_G^G \in \mathbf{e}_K(G) - \mathbf{j}_K(G)$, we have $e_G^G \in \mathbf{e}_H^{\Delta,p}(G)$, and $\beta(G) \cong K$.

Since $e_G^G \in e_H^{\Delta,p}(G)$, there exists a subgroup L of $G \times H$ such that

$$\tilde{e}_G^G\big((G \times H)/L\big)\, e_H^H \neq 0,$$

and such that $k_1(L)$ and $k_2(L)$ are p'-groups. It follows that $p_1(L) = G$ and $p_2(L) = H$. Moreover,

$$\mathrm{Def}^H_{H/k_2(L)}(e_H^H) \neq 0.$$

Since $k_2(L)$ is a p'-group and since H is a $B^{\Delta,p}$-group, this implies $k_2(L) = 1$. Therefore, $H \cong G/k_1(L)$. But since $\beta(G) \cong \beta(H) \cong K$, we have $m_{G,k_1(L)} \neq 0$, and then since $k_1(L)$ is a p'-group, we have $\delta_p(G) \cong \delta_p(H) = H$. Therefore, $\mathbf{e}_H^{\Delta,p} = \mathbf{e}_G^{\Delta,p}$, and so $F = \mathbf{e}_H^{\Delta,p}$.

This shows that the image of $\mathbf{e}_H^{\Delta,p}$ in $\mathbf{e}_K/\mathbf{j}_K$ is simple, and isomorphic to $S_{H,\mathbb{C}}^{\Delta,p}$. It follows that

$$\bigoplus_{\substack{H:\, B^{\Delta,p}\text{-group}\\ \beta(H)\cong K\\ \text{up to iso.}}} S_{H,\mathbb{C}}^{\Delta,p}$$

is a subfunctor of $S_{K,\mathbb{C}}$. Now, since any composition factor of $S_{K,\mathbb{C}}$ isomorphic to $S_{H,\mathbb{C}}^{\Delta,p}$ is generated by the image of e_H^H in $S_{K,\mathbb{C}}(H)$, the simple functor $S_{H,\mathbb{C}}^{\Delta,p}$ has multiplicity one in $S_{K,\mathbb{C}}$. This completes the proof. □

9.3 Corollary *Let $q \neq p$ be a prime. We have the following isomorphisms of p-bifree biset functors.*

(a)

$$S_{1,\mathbb{C}} \cong \bigoplus_{\substack{P:\, \text{cyclic } p\text{-group}\\ \text{up to iso.}}} S_{P,\mathbb{C}}^{\Delta,p}.$$

(b)

$$S_{C_p\times C_p,\mathbb{C}} \cong \bigoplus_{\substack{P:\, \text{non-cyclic } p\text{-group}\\ \text{up to iso.}}} S_{P,\mathbb{C}}^{\Delta,p}.$$

(c)

$$S_{C_q\times C_q,\mathbb{C}} \cong \bigoplus_{\substack{D:\, \text{cyclic } p\text{-group}\\ \text{up to iso.}}} S_{C_q\times C_q\times D,\mathbb{C}}^{\Delta,p}.$$

Proof (a) If H is a finite group, then by [Bc10, Proposition 5.6.1], $\beta(H) = 1$ if and only if H is cyclic. A cyclic group is a $B^{\Delta,p}$-group if and only if it is a cyclic p-group. Theorem 9.2 implies the result.

(b) If H is a finite group, then by [Bc23, Lemma 2.4], $\beta(H) \cong C_p \times C_p$ if and only if $H = P \times C$, where P is a non-cyclic p-group and C is a cyclic p'-group. Then $\delta_p(H) = P$. The result follows from Theorem 9.2.

(c) If H is a finite group, then by [Bc23, Lemma 2.4], $\beta(H) \cong C_q \times C_q$ if and only if $H = Q \times C$, where Q is a non-cyclic q-group and C is a cyclic q'-group. Then $C = D \times E$, where D is a cyclic p-group and E is a cyclic p'-group. Then $H = Q \times D \times E$, and $\delta_p(H) \cong C_q \times C_q \times D$. The result follows from Theorem 9.2. ◻

We now give a proof of the decompositions in Corollary 9.3 via the restriction-kernel technique. We treat the case of the functor $S_{C_p\times C_p,\mathbb{C}}$, the argument for Part (c) is analogous.

9.4 Proposition (a) *Let G be a finite group. One has $\underline{S_{C_p\times C_p,\mathbb{C}}}(G) = \{0\}$ unless G is a non-cyclic p-group.*

(b) *Let G be a non-cyclic p-group. Then $\underline{S_{C_p\times C_p,\mathbb{C}}}(G)$ is one-dimensional generated by the image of e_G^G in $S_{C_p\times C_p,\mathbb{C}}(G)$.*

Proof First note that for any p-bifree biset functor S and finite group G, one has $\underline{S}(G) \subseteq \tilde{e}_G^G S(G)$. Indeed, one has $S(G) = \tilde{e}_G^G S(G) \oplus (1-\tilde{e}_G^G)S(G)$ and $(1-\tilde{e}_G^G)S(G) \cap \underline{S}(G) = \{0\}$.

Now note that by [Bc23, Corollary 3.6], if $\tilde{e}_G^G S_{C_p\times C_p,\mathbb{C}}(G)$ is non-zero, then G is p-elementary. Furthermore, by the proof of [Bc10, Theorem 5.5.4], $\tilde{e}_G^G S_{C_p\times C_p,\mathbb{C}}(G)$ is generated by the image $\overline{e_G^G}$ in $S_{C_p\times C_p,\mathbb{C}}(G)$ of e_G^G if G is not cyclic and is equal to zero if G is cyclic.

Suppose that $G = P \times C$ is a non-cyclic p-elementary group where $P = O_p(G)$. Then $e_G^G = e_P^P \times e_C^C$. Also if $N \trianglelefteq G$ is of p'-order, then $G/N \cong P \times C/N$ and

$$\mathrm{Def}^G_{G/N} e_G^G = e_P^P \times \mathrm{Def}^C_{C/N} e_C^C\,.$$

Since C is a cyclic group, $m_{C,C} \neq 0$ by [Bc10, Proposition 5.6.1] and hence

$$\mathrm{Def}^G_{G/C} e_G^G = m_{C,C}\, e_P^P \times e_{C/C}^{C/C} \neq 0\,.$$

This shows that $\underline{S_{C_p\times C_p,\mathbb{C}}}(G) = \{0\}$ unless $C = 1$. On the other hand if $C = 1$, then $\underline{S_{C_p\times C_p,\mathbb{C}}}(G) = \tilde{e}_G^G S_{C_p\times C_p,\mathbb{C}}(G)$ is one-dimensional generated by $\overline{e_G^G}$. In fact, in this case G is a p-group and $\mathrm{Res}^G_H e_G^G = 0$ for any proper subgroup $H < G$ by [Bc10, Theorem 5.2.4]. This proves both parts. ◻

9.5 Proposition *Let G be a finite group. Then for any $x \in S_{C_p\times C_p,\mathbb{C}}(G)$, there exists $x' \in \underline{S_{C_p\times C_p,\mathbb{C}}}(G)$ and $\alpha \in I_G$ such that $x = x' + \alpha \cdot x$.*

Proof Let $x \in S_{C_p\times C_p,\mathbb{C}}(G)$. By the proof of Proposition 9.4, we have $\tilde{e}_G^G S_{C_p\times C_p,\mathbb{C}}(G) = \{0\}$ if G is not a non-cyclic p-group. In this case, we have $x' = 0$ and $\alpha = 1 - \tilde{e}_G^G$. If G is a non-cyclic p-group, then again by the proof of Proposition 9.4, we have $\underline{S_{C_p\times C_p,\mathbb{C}}}(G) = \tilde{e}_G^G S_{C_p\times C_p,\mathbb{C}}(G)$. Thus, in this case, we choose $x' = \tilde{e}_G^G \cdot x$ and $\alpha = 1 - \tilde{e}_G^G$. □

These two results allow us to determine the full decomposition of $S_{C_p\times C_p,\mathbb{C}}$ as a p-bifree biset functor. It splits as a direct sum of simple functors indexed by non-cyclic p-groups, each occurring with multiplicity one.

9.6 Theorem *One has an isomorphism*

$$S_{C_p\times C_p,\mathbb{C}} \cong \bigoplus_{P:\text{non-cyclic } p\text{-group}} S_{P,\mathbb{C}}^{\Delta,p}$$

of p-bifree biset functors.

Proof By Propositions 9.4 and 9.5 and Theorem 10.2, the composition factors of $S_{C_p\times C_p,\mathbb{C}}$ are exactly the functors $S_{P,\mathbb{C}}^{\Delta,p}$, where P is a non-cyclic p-group, each with multiplicity one. We will show that each of these functors is a subfunctor.

Let F be a p-bifree biset subfunctor of $S := S_{C_p\times C_p,\mathbb{C}}$ and let G be a minimal group of F. Then we have $\{0\} \neq F(G) \subseteq \underline{S}(G)$ which by Proposition 9.4 implies that G is a non-cyclic p-group and that $F(G)$ is one-dimensional generated by $\overline{e_G^G}$. Conversely, if G is a non-cyclic p-group, then the subfunctor generated by $\underline{S}(G)$ has a minimal group G.

For a non-cyclic p-group P, let F_P be the subfunctor of S generated by $\overline{e_P^P}$. Then F_P has a minimal group P. Moreover, if P' is a p-group such that $F_P(P') \neq 0$, then P is isomorphic to a subgroup of P'. Indeed, if X is a transitive p-bifree (P', P)-biset such that $X \circ e_P^P \neq 0$, then $p_2(X) = P$ and $k_2(X) = 1 = k_1(X)$.

We claim that F_P is a simple p-bifree biset functor isomorphic to $S_{P,\mathbb{C}}^{\Delta,p}$. To prove the claim, it suffices to prove that if P' is a non-cyclic p-group not isomorphic to P, then $F_{P'} \not\subseteq F_P$. Indeed, if M is a proper nonzero subfunctor of F_P, then the minimal group of M is a non-cyclic p-group P' non-isomorphic to P. But then $F_{P'} \subseteq M$, since $M(P')$ is one dimensonal generated by $\overline{e_{P'}^{P'}}$. Now, as above, $F_P(P') = 0$ if P is not isomorphic to a subgroup of P'. Also, if P is isomorphic to a subgroup of P', then $F_P(P')$ is one-dimensional generated by $\mathrm{Ind}_P^{P'} e_P^P$, or by $e_P^{P'}$. In both cases, it follows that if $P \not\cong P'$, then $e_{P'}^{P'} F_P(P') = 0$. But $e_{P'}^{P'} F_{P'}(P') = F_{P'}(P') \neq 0$. Thus, $F_{P'} \not\subseteq F_P$ and hence F_P is simple. Since $F_P(P)$ is one-dimensional and generated by $\overline{e_P^P}$ which is invariant under any automorphism of P, we have $F_P(P) \cong \mathbb{C}$. Therefore, $F_P \cong S_{P,\mathbb{C}}^{\Delta,p}$. This proves the theorem. □

10 Appendix

Let R be a commutative ring and let $\mathbb{K}$ be a field.

10.1 Lemma *Let F be a (p-bifree) biset functor over R and G a finite group. Let $\mathrm{Sec}(G)$ be the set of all proper subquotients of G. Then we have*

$$\underline{F}(G) = \bigcap_{\substack{H \in \mathrm{Sec}(G) \\ \alpha \in RB^{\Delta}(H,G)}} \ker(F(\alpha) : F(G) \to F(H))\,.$$

Proof Denote the right hand side of the above equality by $\underline{\underline{F}}(G)$. Clearly $\underline{F}(G) \subseteq \underline{\underline{F}}(G)$.

For the reverse inclusion, let T be a group of order less than $|G|$ and let $U \leqslant T \times G$. Also let $x \in \underline{\underline{F}}(G)$. It is sufficient to prove that

$$\Big(\frac{T \times G}{U}\Big) \cdot x = 0.$$

Write

$$\Big(\frac{T \times G}{U}\Big) = \mathrm{Ind}_P^T \mathrm{Inf}_{P/K}^T \mathrm{Iso}(f) \mathrm{Def}_{Q/L}^Q \mathrm{Res}_Q^G$$

with the usual choices of letters. Thus

$$\Big(\frac{T \times G}{U}\Big) \cdot x = (\mathrm{Ind}_P^T \mathrm{Inf}_{P/K}^T \mathrm{Iso}(f) \mathrm{Def}_{Q/L}^Q \mathrm{Res}_Q^G) \cdot x = \mathrm{Ind}_P^T \mathrm{Inf}_{P/K}^T \mathrm{Iso}(f) (\mathrm{Def}_{Q/L}^Q \mathrm{Res}_Q^G \cdot x) = 0.$$

Here $\mathrm{Def}_{Q/L}^Q \mathrm{Res}_Q^G \cdot x = 0$ since $x \in \underline{\underline{F}}(G)$. □

The following theorem is stated in [BCK] for Green biset functors. But one can easily check that it is also valid for p-bifree biset functors. We include the proof for the sake of self-containment.

10.2 Theorem *[BCK, Theorem 2.5] Let F be a p-bifree biset functor over a field k and let G be a finite group. Suppose that $\dim_k F(G) < \infty$, and that for every $\mathcal{E}_G$-submodule $M \subseteq F(G)$, one has*

$$M = (M \cap \underline{F}(G)) + I_G M. \qquad (*)$$

Then, for every simple $k[\mathrm{Out}(G)]$-module V, the following numbers are equal:

1. *Multiplicity $[F : S_{G,V}]$ of $S_{G,V}$ as a composition factor of F.*
2. *Multiplicity of V as a composition factor of the $\mathcal{E}_G$-module $F(G)$.*
3. *Multiplicity of V as a composition factor of the $k[\mathrm{Out}(G)]$-module $\underline{F}(G)$.*

Proof The equality of the first two numbers are well-known. We prove the equality of the last two numbers. Since $\dim_k F(G) < \infty$, there exists an $\mathcal{E}_G$-composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = F(G)$$

of $F(G)$. Set $K := \underline{F}(G)$, and consider the induced series

$$0 = (M_0 \cap K) \subseteq (M_1 \cap K) \subseteq \cdots \subseteq (M_{n-1} \cap K) \subseteq (M_n \cap K) = K$$

of $\mathcal{E}_G$-submodules of K. Let V be a simple $k[\mathrm{Out}(G)]$-module, and let $i \in \{1, \ldots, n\}$. We claim that if $M_i/M_{i-1} \cong V$, then $(M_i \cap K)/(M_{i-1} \cap K) \cong V$. This implies that $[F(G) : V] \leq [K : V]$. But clearly, $[K : V] \leq [F(G) : V]$, and we obtain equality.

To prove the claim, observe that

$$\frac{M_i \cap K}{M_{i-1} \cap K} = \frac{M_i \cap K}{(M_i \cap K) \cap M_{i-1}} \cong \frac{(M_i \cap K) + M_{i-1}}{M_{i-1}} \subseteq \frac{M_i}{M_{i-1}} \cong V.$$

It therefore suffices to show that the left-hand side is nonzero. Assume by contradiction that $M_i \cap K = M_{i-1} \cap K$. Then,

$$M_i = (M_i \cap K) + I_G M_i \subseteq (M_i \cap K) + M_{i-1} = M_{i-1},$$

contradicting $M_i/M_{i-1} \cong V \neq 0$. Here, we used the assumption that V is annihilated by I_G, and the hypothesis of the theorem. □

10.3 Remark Let F be a p-bifree biset functor over $\mathbb{K}$ and let G be a finite group. It is straightforward to prove that the condition in the above theorem is equivalent to any of the following conditions. We include the proof of the equivalences from [BCK, Proposition 2.6] for completeness.

(1) For every subfunctor M of F one has $M(G) = \underline{M}(G) + \mathcal{J}M(G)$.

(2) For every $x \in F(G)$, there exist $x' \in \underline{F}(G)$ and $\alpha \in I_G$ with $x = x' + \alpha \cdot x$.

Proof of equivalences: Suppose Condition (1) holds and let $x \in F(G)$. Set $M := \langle x \rangle$ to be the subfunctor of F generated by x. Then, by definition, $\mathcal{J}M(G) = I_G \cdot x$, hence (2) holds.

Next suppose (2) holds and let M be an $\mathcal{E}_G$-submodule of $F(G)$. Also let $x \in M$. Writing x as $x = x' + \alpha x$, by (2), we see that $x' = x - \alpha x \in M$. Hence $M \subseteq (M \cap \underline{F}(G)) + I_G M$. The reverse inclusion is trivial. Hence the condition $(*)$ of Theorem 10.2 holds.

Finally, suppose $(*)$ holds and let M be a subfunctor of F. Then it is easy to see that $M(G)$ is an $\mathcal{E}_G$-submodule of $F(G)$, that $\underline{M}(G) = M(G) \cap \underline{F}(G)$ and that $I_G M(G) \subseteq \mathcal{J}M(G)$. Hence (1) holds.

10.4 Proposition *Let F be a semisimple p-bifree biset functor. For any finite group G, we have*

$$F(G) = \underline{F}(G) \oplus \mathcal{J}F(G).$$

In particular, for every $x \in F(G)$, there exist $x' \in \underline{F}(G)$ and $\alpha \in I_G$ satisfying

$$x = x' + \alpha \cdot x\,.$$

Proof Let $n_{H,V}$ denote the multiplicity of the simple p-bifree biset functor $S_{H,V}^{\Delta,p}$ in F. Then evaluating at G, we have

$$F(G) \cong \left(\bigoplus_{W} n_{G,W} S_{G,W}^{\Delta,p}(G) \right) \oplus \left(\bigoplus_{(H,V):|H|<|G|} n_{H,V} S_{H,V}^{\Delta,p}(G) \right) .$$

Using arguments similar to the proof of Proposition 8.5, one can show that the first summand is equal to $\underline{F}(G)$ and that the second summand is equal to $\mathcal{J}F(G)$. This proves the first assertion. The second assertion follows from the above remark. □

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