MAXIMAL DEFECT 2-BLOCKS WITH DIHEDRAL DEFECT GROUPS UP TO SPLENDID MORITA EQUIVALENCE

SHIGEO KOSHITANI, CAROLINE LASSUEUR, İPEK TUVAY, DENIZ YILMAZ

Abstract. We classify, up to splendid Morita equivalence, all maximal defect blocks whose defect groups are dihedral 2-groups of order at least 8. The classification holds for blocks over an algebraically closed field as well as for those over a suitable discrete valuation ring of characteristic zero. As a consequence, Puig's conjecture holds for this infinite family of blocks.

Beyond the classification result itself, a central feature of the present work is the variety of methods required in its proof. Our arguments combine classical techniques of block theory with a broad range of tools and techniques that have emerged over the past twenty-five years. Amongst others these include in a crucial manner the Gorenstein-Walter classification of finite groups with dihedral Sylow 2-subgroups, Linckelmann's glueing of local Morita equivalences to obtain a stable equivalence of Morita type, the modern versions of Fong's 1st and 2nd reduction theorems in terms of splendid and endo-permutation source Morita equivalences, the structure of the Dade group of endo-permutation modules, Boltje-Perepelitsky's extended theory of Brauer pairs for p-permutation modules, the fusion systems of the blocks involved, or the concrete knowledge of their decomposition matrices.

1. Introduction

Let p be a prime number, let k be an algebraically closed field of characteristic p and let G be a finite group. An indecomposable direct algebra factor of the group algebra kG is called a *block* (or *block algebra*, or a *p-block* of G). The present work is part of the general programme of classifying blocks up to certain categorical equivalences, which preserve various representation-theoretic invariants.

Two notions of equivalence play a central role in this programme: the standard notion of *Morita equivalence* of finite-dimensional k-algebras and the stronger notion of *splendid Morita equivalence*. Donovan's Conjecture and Puig's Finiteness Conjecture predict, respectively, that there are only finitely many Morita equivalence classes and splendid Morita equivalence classes p-blocks with a given isomorphism type of defect groups. A proof of Donovan's Conjecture has recently been claimed by OpenAI [Ope26a, Ope26b]. Beyond these finiteness statements, a natural, and much more involved and deeper task than their numerical counting is to classify the equivalence classes explicitly.

Date: November 27, 2026.

2020 *Mathematics Subject Classification.* Primary 20C05, 20C15, 20C20, 20C30, 20C33. Secondary 16D20, 16D90.

Key words and phrases. Block theory, dihedral 2-groups, Puig's Conjecture, splendid Morita equivalences, Morita equivalences, stable equivalences of Morita type, p-permutation modules, trivial source modules, source algebras, Fong's reduction theorems.

The authors gratefully acknowledge the generous support and hospitality of the Mathematisches Forschungsinstitut Oberwolfach, where this project developed in January 2026 as part of the Oberwolfach Research Fellowship Programme for research in small groups.

The fast-growing database of blocks maintained by Charles Eaton [Eat26] provides an excellent overview of the progress made towards classifying blocks up to Morita equivalence. Much less is known about their classification up to splendid Morita equivalence. Puig's Finiteness Conjecture holds for blocks with cyclic defect groups, for any prime number p [Lin96a], and for blocks with Klein-four defect groups when $p = 2$ [CEKL11]. In the latter case, [CEKL11] also provides us with a complete classification: there are three splendid Morita equivalence classes. For cyclic defect groups, however, a complete classification is not yet known, even up to Morita equivalence. A good summary of the known/unknown embedded Brauer trees can be found in [Cra19, Chapter 5], while [HL24, HL25, HL26] investigate one of the further invariants needed to determine the source algebra of such blocks.

These considerations illustrate the difficulty of classifying blocks with a given defect group up to splendid Morita equivalence, even when their representation theory is well-understood. Nevertheless, further finiteness results in the direction of Puig's Finiteness Conjecture have been obtained by restricting the class of finite groups, respectively the class of blocks, considered. These include [Pui88] for all blocks of p-soluble groups, [Pui94, Kes96] for all blocks of the symmetric groups and their double covers, [Kes96, Kes02] for all blocks of the alternating groups and their double covers, [Kes00] for all blocks of Weyl groups, [HK00, HK05, Kes01] for the unipotent blocks of the finite classical groups, and [KL19, KL20, KLS21] for principal blocks of tame representation type.

In this spirit, a natural next step is to investigate arbitrary 2-blocks of tame representation type, i.e. with dihedral, semi-dihedral or generalized quaternion defect groups. In this article, we take a first step in this direction by considering tame blocks of maximal defect with defect groups isomorphic to a dihedral 2-group D_{2^n} of order 2^n with $n \geq 3$. Our main result shows that every such block is splendidly Morita equivalent to a principal block.

Theorem 1.1. *Let k be an algebraically closed field of characteristic 2 and let G be a finite group with a dihedral Sylow 2-subgroup of order 2^n ($n \geq 3$). Then, any block $\mathbf{B}$ of kG with maximal defect is splendidly Morita equivalent to precisely one of the following blocks:*

(a) $\mathbf{B}_0(kD_{2^n})$*;*
(b) $\mathbf{B}_0(k\mathfrak{A}_7)$ *(provided $n = 3$);*
(c) $\mathbf{B}_0(k\mathrm{PSL}_2(q))$ *where q is an odd prime power such that $(q-1)_2 = 2^n$;*
(d) $\mathbf{B}_0(k\mathrm{PSL}_2(q))$ *where q is an odd prime power such that $(q+1)_2 = 2^n$;*
(e) $\mathbf{B}_0(k\mathrm{PGL}_2(q))$ *where q is an odd prime power such that $(q-1)_2 = 2^{n-1}$;*
(f) $\mathbf{B}_0(k\mathrm{PGL}_2(q))$ *where q is an odd prime power such that $(q+1)_2 = 2^{n-1}$.*

In cases (c)–(f)*, any choice of q satisfying the indicated condition yields a representative of the corresponding splendid Morita equivalence class.*

We recall that Erdmann classified tame k-algebras up to Morita equivalence through an explicit list of families of basic algebras, defined by quivers and relations; see [Erd90]. This classification was recently refined by Macgregor [Mac22] using the classification of the finite simple groups, giving a complete list of Morita equivalence classes of blocks of group algebras with dihedral defect groups. Theorem 1.1 shows that the list obtained in [Mac22] does not split further up to splendid Morita equivalence if we restrict ourselves to blocks of maximal defect.

Now, if we consider a complete discrete valuation ring $\mathcal{O}$ of characteristic zero with residue field $\mathcal{O}/J(\mathcal{O}) = k$ (our previously fixed algebraically closed field of characteristic 2), then the results of Eisele in [Eis12] prove that Erdmann's and Macgregor's classification can be lifted to $\mathcal{O}$. Adding the splendid condition, Theorem 1.1 can be lifted to $\mathcal{O}$ as a splendid Morita equivalence given by tensoring with a 2-permutation bimodule, which is liftable. Moreover,

Theorem 1.1 directly implies that Puig's finiteness conjecture holds when we restrict ourselves to the class of all blocks of all finite groups with defect groups isomorphic to a fixed dihedral 2-group and maximal defect.

Corollary 1.2. *For any $n \geq 3$, there are only finitely many splendid Morita equivalence classes of blocks of the group algebra of finite groups, over k and over $\mathcal{O}$, with defect groups isomorphic to D_{2^n} and maximal defect.*

We emphasize that beyond the classification result of Theorem 1.1 itself, a significant aspect of the present work lies in the wide range of methods and techniques that are needed for its proof. The arguments combine many classical tools from group and block theory with many more recent results and methods that have emerged over the past 25 years. See the abstract. Moreover, the dihedral case considered here will also serve as a prototype for the treatment of the remaining tame cases, that is, the cases of generalized quaternion and semi-dihedral defect group, which we will treat in forthcoming papers. The recent work of Eaton, Eisele, Kessar, Linckelmann and Schaeffer Fry [EEK+26] classifying 2-blocks with generalized quaternion defect groups up to Morita equivalence will be of substantial help in the treatment of this case.

We also emphasize that the main difficulty at the heart of the proof lies in the problem of establishing splendid Morita equivalences between the faithful 2-blocks of maximal defect of $3.\mathfrak{A}_6$ and $3.\mathfrak{A}_7$ and suitable principal blocks.

The paper is organized as follows. To begin with, in Section 2 and 3, we introduce the background material which is essential for the proof of Theorem 1.1. In Section 4, we perform a reduction to $G/O_{2'}(G)$ and its central extensions by a certain odd cyclic group. In Section 5 and 6, we deal with four isolated cases arising from the exceptional covering groups $3.\mathfrak{A}_6$ and $3.\mathfrak{A}_7$. Finally, we prove our main results in Section 7. Furthermore, in Appendix A we give a proof that Fong's second reduction theorem gives a Morita equivalence with endo-permutation source and diagonal vertex.

2. Notation, prerequisites and quoted results

2.1 General notation. Throughout this manuscript, unless otherwise specified, we adopt the following notation and conventions.

All groups considered are finite, and G and H always denote finite groups. We let k be an algebraically closed field of characteristic $p > 0$. From Section 3 on, we assume that $p = 2$. When character-theoretic arguments are needed, we let $\mathcal{O}$ be a complete discrete valuation ring with residue field $\mathcal{O}/J(\mathcal{O}) = k$ and a quotient field K of characteristic zero, which is assumed to be large enough for the finite groups under consideration and their subgroups.

For $R \in \{k, \mathcal{O}\}$ all modules over a given R-algebra $\mathbf{A}$ that is free of finite rank over R, are assumed to be finitely generated left modules and we denote by $\mathbf{A} - \mathsf{mod}$ the corresponding module category. The set of all blocks of $\mathbf{A}$ (i.e. its indecomposable algebra factors) is denoted $\mathrm{Bl}(\mathbf{A})$. We write $\mathbf{B}_0(RG)$ for the principal block of RG. Given $\mathbf{B} \in \mathrm{Bl}(RG)$, we write $e_{\mathbf{B}}$ for the block idempotent of RG corresponding to $\mathbf{B}$, and if $D \leq G$ is a defect group of $\mathbf{B}$, then the integer d such that $|D| = p^d$ is called the defect of $\mathbf{B}$. We say that $\mathbf{B}$ is a block of *maximal defect* provided its defect groups are the Sylow p-subgroups of G, i.e. $|G|_p = p^d$, and we let $\mathrm{Bl}^{\max}(RG)$ denote the subset of $\mathrm{Bl}(RG)$ consisting of all blocks of maximal defect of RG. If $N \trianglelefteq G$ and $\mathbf{c} \in \mathrm{Bl}(RN)$, then $\mathrm{Bl}(RG \mid \mathbf{c}) := \{\mathbf{B} \in \mathrm{Bl}(RG) \mid \mathbf{B} \text{ covers } \mathbf{c}\}$.

The canonical algebra homomorphism $\mathcal{O}G \longrightarrow kG$ induces a defect group preserving bijection between $\mathrm{Bl}(\mathcal{O}G)$ and $\mathrm{Bl}(kG)$, mapping a block $\mathbf{B} \in \mathrm{Bl}(\mathcal{O}G)$ to $k \otimes_{\mathcal{O}} \mathbf{B}$. In this context, we write $\mathrm{Irr}(\mathbf{B})$, and also simply $\mathrm{Irr}(k \otimes_{\mathcal{O}} \mathbf{B})$ when working over k, for the sets of all irreducible

K-characters of G belonging to $\mathbf{B}$. We write $\mathrm{Irr}(G)$ for the set of all irreducible K-characters of G.

In contrast, if $\mathbf{B} \in \mathrm{Bl}(kG)$, then $\mathrm{Irr}_k(\mathbf{B})$ denotes the set of all simple $\mathbf{B}$-modules up to isomorphism. We denote by k_G, or simply k, the trivial kG-module. For $H \leq G$ we denote by Res^G_H (resp. Ind^G_H) the restriction (resp. the induction) of kG-modules and characters. Similarly, for $N \trianglelefteq G$, we denote by $\mathrm{Inf}^G_{G/N}$ the inflation of kG-modules and characters. Given a p-subgroup $Q \leq G$ and a kG-module M we denote by $M[Q]$ the Brauer construction of M with respect to Q, and by Br^G_Q (or simply Br_Q when clear from the context) the Brauer homomorphism with respect to Q, i.e. the surjective algebra homomorphism $\mathrm{Br}^G_Q : (kG)^Q \longrightarrow kC_G(Q)$ mapping $C_G(Q)$ to itself by the identity map and mapping all the other group elements to zero. The projective cover of a simple kG-module S is written $P(S)$. If S_1, S_2 are simple kG-modules, then $U(S_1, S_2)$ denotes a uniserial kG-module with head S_1 and socle S_2.

Again, given $R \in \{\mathcal{O}, k\}$, an RG-lattice M is a *permutation* RG-lattice if it admits a G-invariant R-basis, whereas it is a *p-permutation* RG-lattice if it is a direct summand of a permutation RG-lattice. Indecomposable p-permutation RG-lattices are also called *trivial source* RG-lattices. It is well known that any p-permutation kG-module M is liftable to an $\mathcal{O}G$-lattice. In general, such modules afford several lifts, but there is a unique one amongst these which is a p-permutation $\mathcal{O}G$-lattice. We denote this lift by $\widehat{M}$ and we let $\chi_{\widehat{M}}$ be the character afforded by $K \otimes_{\mathcal{O}} \widehat{M}$. For any p-subgroup Q of G, the Scott module with vertex Q, written $\mathrm{Sc}(G, Q)$, is always a trivial source module.

Given an integer $n \geq 2$, C_n denotes the cyclic group of order n, $\mathfrak{A}_n$ and $\mathfrak{S}_n$ the alternating and the symmetric group on n letters, respectively, and D_{2^n} the dihedral group of order 2^n. Furthermore, $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G, and for $P \in \mathrm{Syl}_p(G)$ we let $\mathcal{F}_P(G)$ be the fusion system of G on P. If $\varphi : Q \longrightarrow P$ is an isomorphism between subgroups $Q \leq H$ and $P \leq G$, then $\Delta(P, \varphi, Q) := \{(\varphi(y), y) \in P \times Q \mid y \in Q\}$ is a twisted diagonal subgroup of $G \times H$, and if $G = H$ and $P = Q$ then $\Delta(P) := \Delta(P, \mathrm{Id}, P)$ is the diagonal embedding of P in $G \times G$. Further, we denote the (external) central product of two finite groups G_1 and G_2 identified along a group isomorphism $\varphi : Z_1 \longrightarrow Z_2$ with $Z_i \leq Z(G_i)$ for $i = 1, 2$ by $G_1 * G_2$, so $G_1 * G_2 = (G_1 \times G_2)/Z$ with $Z = \{(z, \varphi(z)^{-1}) \mid z \in Z_1\} \leq Z(G_1 \times G_2)$.

For further standard notation and terminology, we refer the reader to [Lin18a, Lin18b, NT89, Thé95].

2.2 Definitions: Morita equivalences, stable equivalences of Morita type, splendid Morita equivalences and endo-permutation source Morita equivalences.
Let $\mathbf{A}$ and $\mathbf{B}$ be finite-dimensional R-algebras with $R \in \{\mathcal{O}, k\}$. We recall that a pair (M, N) consisting of an $(\mathbf{A}, \mathbf{B})$-bimodule M and of a $(\mathbf{B}, \mathbf{A})$-bimodule N is said to *induce a Morita equivalence between* $\mathbf{A}$ *and* $\mathbf{B}$ if M and N are finitely generated projective as left and right modules, and if there are isomorphisms $M \otimes_{\mathbf{B}} N \cong \mathbf{A}$ and $N \otimes_{\mathbf{A}} M \cong \mathbf{B}$ of $(\mathbf{A}, \mathbf{A})$-bimodules and $(\mathbf{B}, \mathbf{B})$-bimodules respectively. In this case, the functors $M \otimes_{\mathbf{B}} - : \mathbf{B} - \mathrm{mod} \longrightarrow \mathbf{A} - \mathsf{mod}$ and $N \otimes_{\mathbf{A}} - : \mathbf{A} - \mathrm{mod} \longrightarrow \mathbf{B} - \mathsf{mod}$ are inverse equivalences of categories, and $\mathbf{A}$ and $\mathbf{B}$ are called *Morita equivalent*. Moreover, as bimodules, $N \cong \mathrm{Hom}_{\mathbf{A}}(M, \mathbf{A})$ and $M \cong \mathrm{Hom}_{\mathbf{B}}(N, \mathbf{B})$, so we often simply say that M *induces a Morita equivalence between* $\mathbf{A}$ *and* $\mathbf{B}$. See [Lin18a, Theorem 2.8.2]. Under the weaker assumptions that $M \otimes_{\mathbf{B}} N \cong \mathbf{A} \oplus$ (projective $(\mathbf{A}, \mathbf{A})$-bimodule) and $N \otimes_{\mathbf{A}} M \cong \mathbf{B} \oplus$ (projective $(\mathbf{B}, \mathbf{B})$-bimodule), it is said that the pair (M, N) induces a *stable equivalence of Morita type* between $\mathbf{A}$ and $\mathbf{B}$.

Assume now that G and H are finite groups, $\mathbf{A} \in \mathrm{Bl}(RG)$, and $\mathbf{B} \in \mathrm{Bl}(RH)$. Then, the $(\mathbf{A}, \mathbf{B})$-bimodule M can be regarded as a left $R[G \times H]$-module (via the left action given by $(g, h) \cdot m := g \cdot m \cdot h^{-1}$ for any $(g, h) \in G \times H$ and any $m \in M$) and the $(\mathbf{B}, \mathbf{A})$-bimodule N as a left

$R[H \times G]$-module. The Morita equivalence is called a *splendid Morita equivalence* (respectively, an *endo-permutation source Morita equivalence*) if M and N, seen as left modules, are trivial source modules (respectively, have an endo-permutation source). Clearly, any splendid Morita equivalence is an endo-permutation source Morita equivalence. If there is an endo-permutation source Morita equivalence between $\mathbf{A}$ and $\mathbf{B}$, then they have isomorphic defect groups. After identifying defect groups of $\mathbf{A}$ and $\mathbf{B}$ with a p-group D, the modules M and N have vertex ΔD. A splendid Morita equivalence is sometimes also called a *Puig equivalence* or a *source-algebra equivalence*, since it is equivalent to the existence of an isomorphism between source algebras of $\mathbf{A}$ and $\mathbf{B}$ as interior D-algebras. See [Lin18b, Proposition 9.7.1, Remark 9.7.2, Definition 9.7.3, Theorem 9.7.4 and Theorem 9.11.2].

2.3 Blocks of direct and central products. Let G and H be finite groups as in the previous subsection. Recall that there is a canonical R-algebra isomorphism $R[G \times H] \longrightarrow RG \otimes_R RH$, mapping $(g, h) \in G \times H$ to $g \otimes h$. Identifying $R[G \times H]$ and $RG \otimes_R RH$ via this isomorphism, each block $\mathbf{B} \in \mathrm{Bl}(R[G \times H])$ can be written as $\mathbf{B} = \mathbf{B}_G \otimes_R \mathbf{B}_H \in \mathrm{Bl}(RG \otimes_R RH)$ with $\mathbf{B}_G \in \mathrm{Bl}(RG)$ and $\mathbf{B}_H \in \mathrm{Bl}(RH)$.

More generally, if $G * H$ is a central product of G and H identified along a central subgroup of p'-order, then any block $\mathbf{B} \in \mathrm{Bl}(R[G * H])$ is isomorphic to $\mathbf{B}_G \otimes_R \mathbf{B}_H$ where $\mathbf{B}_G \in \mathrm{Bl}(RG)$ is the unique block of RG covered by $\mathbf{B}$ and $\mathbf{B}_H \in \mathrm{Bl}(RH)$ is the unique block of RH covered by $\mathbf{B}$. See [Düv04, Proposition 1.5] (or also [Sam14, Lemma 7.5]). More precisely, the block $\mathbf{B}_G \otimes_R \mathbf{B}_H$, viewed as a block of $R[G \times H]$, maps onto $\mathbf{B}$ under the canonical map $R[G \times H] \to R[G * H]$. By [Düv04, Lemma 1.4], this map induces an algebra isomorphism

$$\mathbf{B}_G \otimes_R \mathbf{B}_H \cong \mathbf{B}.$$

Since the kernel of the canonical homomorphism $G \times H \to G * H$ is a p'-group, the quotient map identifies the corresponding defect groups. If D denotes the common defect group, then the above map, being induced by a group homomorphism, is compatible with the corresponding interior D-algebra structures. Thus, $\mathbf{B}_G \otimes_R \mathbf{B}_H$ and $\mathbf{B}$ have isomorphic source algebras as interior D-algebras. In particular, they are splendidly Morita equivalent.

Lemma 2.1. *Let $G * H$ be a central product of finite groups identified along a central subgroup of p'-order (possibly trivial). Let $\mathbf{B} \in \mathrm{Bl}(R[G * H])$ and let $\mathbf{B}_G \in \mathrm{Bl}(RG)$ and $\mathbf{B}_H \in \mathrm{Bl}(RH)$ be the unique blocks covered by $\mathbf{B}$, as above. If $\mathbf{B}_G$ is a block of defect zero, then $\mathbf{B}$ and $\mathbf{B}_H$ are splendidly Morita equivalent.*

Proof. By the above discussions, the blocks $\mathbf{B}$ and $\mathbf{B}_G \otimes_R \mathbf{B}_H$ are splendidly Morita equivalent. By [Thé95, p. 341, line -9], R is a source algebra of $\mathbf{B}_G$, thus the assertion follows from [KMN11, Lemma 2.6(i)]. □

2.4 Groups with dihedral Sylow 2-subgroups. Let G be a finite group with a dihedral Sylow 2-subgroup P of order $2^n \geq 8$. Then, Gorenstein and Walter [GW65, Theorem 1] proved that $\overline{G} := G/O_{2'}(G)$ is isomorphic to P, the alternating group $\mathfrak{A}_7$, or a subgroup of $\mathrm{P\Gamma L}_2(q)$ containing $\mathrm{PSL}_2(q)$, where $q := r^f$ is a power of an odd prime r and $f \geq 1$ is an integer. Then, $\mathrm{P\Gamma L}_2(q) \cong \mathrm{PGL}_2(q) \rtimes \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_r)$, where $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_r)$ is cyclic of order f. Thus, for the purposes of our computations, we split the above list of groups into further subcases as follows:

- **(D1)** $\overline{G} \cong D_{2^n}$;
- **(D2)** $\overline{G} \cong \mathfrak{A}_7$;
- **(D3)** $\overline{G} \cong \mathrm{PSL}_2(q) \rtimes C_d$, where q is a power of an odd prime such that $(q-1)_2 = 2^n$ or $(q+1)_2 = 2^n$, d is odd and $d \mid f$;

(D4) $\overline{G} \cong \mathrm{PGL}_2(q) \rtimes C_d$, where q is a power of an odd prime such that $(q-1)_2 = 2^{n-1}$ or $(q+1)_2 = 2^{n-1}$, d is odd and $d \,|\, f$.

Notice that Case **(D3)** happens provided $q \equiv \pm 1 \pmod 8$, whereas Case **(D4)** happens when $q \equiv \pm 1 \pmod 4$. In both cases, d is odd. Moreover, $\mathrm{PSL}_2(9) \cong \mathfrak{A}_6$ and $\mathrm{PGL}_2(9) \cong \mathfrak{A}_6.2_2$ in the ATLAS notation.

Next, recall that the Schur multiplier of a finite group G is the abelian group $\mathrm{H}^2(G, \mathbb{C}^\times)$ and there is an isomorphism

$$(*) \qquad \mathrm{H}^2(G, k^\times) \cong \mathrm{H}^2(G, \mathbb{C}^\times)_{2'}$$

(see e.g. [Kar87, 2.1.14 Proposition]). For the groups above, this group is as follows.

Lemma 2.2. *Assume $p = 2$. Let G be a finite group with a dihedral Sylow 2-subgroup of order 2^n with $n \geq 3$ and assume that $O_{2'}(G) = 1$. Then, we have*

$$\mathrm{H}^2(G, k^\times) \cong \begin{cases} C_3 & \text{if } G \in \{\mathfrak{A}_7, \mathrm{PSL}_2(9)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, if $G \in \{\mathfrak{A}_7, \mathrm{PSL}_2(9)\}$ and Z is a finite cyclic group of odd order on which G acts trivially and $3 \mid |Z|$, then

$$\mathrm{H}^2(G, Z) \cong \mathrm{H}^2(G, k^\times) \cong C_3\,.$$

Proof. We go through the cases **(D1)** to **(D4)**. First, if $G = D_{2^n}$, then $\mathrm{H}^2(G, k^\times) \cong 1$ as G is a 2-group; see e.g. [NT89, Chapter 2, Theorem 7.4].

If $G \in \{\mathfrak{A}_7, \mathrm{PSL}_2(9)\}$, then we read from the ATLAS [CCN$^+$85, p.10 and p.4 resp.] that $\mathrm{H}^2(G, \mathbb{C}^\times) \cong C_6$, so $\mathrm{H}^2(G, k^\times) \cong C_3$ by $(*)$.

If **(D3)** holds with $d = 1$, $G = \mathrm{PSL}_2(q)$ and $q \neq 9$, then $\mathrm{H}^2(G, \mathbb{C}^\times) \cong C_2$ by [CCN$^+$85, Chapter 3, §3 with Table 5], hence $\mathrm{H}^2(G, k^\times) \cong 1$ by $(*)$.

Next, if $G \cong \mathrm{PGL}_2(3) \cong \mathfrak{S}_4$, then we have $\mathrm{H}^2(G, \mathbb{C}^\times) \cong C_2$ by [Kar87, 2.12.3 Theorem], hence $\mathrm{H}^2(G, k^\times) \cong 1$ by $(*)$. If $G = \mathrm{PGL}_2(9)$, then we read from the ATLAS [CCN$^+$85, bottom of p.5 together with page xxii] and $(*)$ that $\mathrm{H}^2(G, k^\times) \cong 1$.

We go through the remaining groups using [Jon74, Theorem 3.1(c)], which states that if $N \trianglelefteq G$ is such that G/N is cyclic, then $|\mathrm{H}^2(G, \mathbb{C}^\times)|$ divides $|\mathrm{H}^2(N, \mathbb{C}^\times)| \cdot |N/[N,N]|$. In all cases, we take $N := \mathrm{PSL}_2(q)$, which is perfect. If G is of type **(D3)** with $d \neq 1$, then $G/N \cong C_d$, and certainly $q > 9$. If G is of type **(D4)** and $q \notin \{3, 9\}$, then $G/N \cong C_{2d}$. In both cases, we obtain that

$$|\mathrm{H}^2(G, \mathbb{C}^\times)| \,\big|\, |\mathrm{H}^2(N, \mathbb{C}^\times)| \cdot |N/[N,N]| = 2 \cdot 1 = 2\,,$$

thus $\mathrm{H}^2(G, k^\times) \cong 1$ by $(*)$. This proves the first claim. The second claim follows from the Universal Coefficient Theorem, which reads

$$\mathrm{H}^2(G, Z) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(G, \mathbb{Z}), Z) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(G, \mathbb{Z}), Z)\,.$$

(See e.g. [Wei94, Exercise 6.1.5].) Since $\mathfrak{A}_7$ and $\mathrm{PSL}_2(9)$ are perfect, the Ext part vanishes, because $\mathrm{H}_1(G, \mathbb{Z}) \cong G/[G,G]$. Moreover, $\mathrm{H}_2(G, \mathbb{Z}) \cong \mathrm{H}^2(G, \mathbb{C}^\times)$ and we obtain $\mathrm{H}^2(G, Z) \cong \mathrm{Hom}_{\mathbb{Z}}(C_3, Z) \cong C_3$. □

2.5 Endo-permutation modules and the Dade group. Let P denote a p-group. A kP-module M is called *endo-permutation* if $\mathrm{End}_k(M)$ is a permutation kP-module. Moreover, such an endo-permutation kP-module M is called *capped* provided it possesses an indecomposable direct summand with vertex P, denoted $\mathrm{cap}(M)$ and called its *cap*. In this case, $\mathrm{cap}(M)$ is unique up to isomorphism, although it may appear with multiplicity greater than one, so $M = \mathrm{cap}(M)^a \oplus X$ with $a \in \mathbb{Z}_{>0}$ and X is a kP-submodule of M, the indecomposable direct summands of which have vertices of order strictly smaller than $|P|$. This allows us to define an equivalence relation: two capped endo-permutation kP-modules are equivalent if and only if their caps are isomorphic. The *Dade group* of P, denoted $\mathbf{D}_k(P)$, is the resulting set of equivalence classes (of capped endo-permutation kP-modules) with group law given by the tensor product $\otimes_k$. By a result of Puig, $\mathbf{D}_k(P)$ is a finitely generated abelian group. We denote its torsion part by $\mathbf{D}_k^{tors}(P)$. For a detailed introduction to endo-permutation modules and their properties, we refer the reader to [Thé07].

We need the next lemma linking sources of simple modules to the structure of the Dade group, the proof of which consists of the arguments given in [CEKL11, paragraph after Proposition 2.1].

Lemma 2.3. *Let G be a p-soluble group. Let T be a simple kG-module, let Q be a vertex of T and let U be a kQ-source of T. Then, the following assertions hold:*

(a) *U is a capped endo-permutation kQ-module and its class in the Dade group is a torsion element, i.e. $[U] \in \mathbf{D}_k^{tors}(Q)$; and*
(b) *if $p = 2$ and $Q \cong D_{2^n}$ with $n \geq 3$, then $U \cong k$.*

Proof. (a) First, U is an endo-permutation module by [Pui88]. Furthermore, by a result of Feit, the simple module T is algebraic; see [Fei80]. Thus, [BK06, Proposition 4.4] proves that the class of U in the Dade group $\mathbf{D}_k(Q)$ must be a torsion element.
(b) As Q is a dihedral 2-group, we have $\mathbf{D}_k^{tors}(Q) = \{[k]\}$ (see e.g. [Thé07, Proposition 6.4(d)]), proving that U is the trivial kQ-module. □

2.6 Clifford-theoretic results. We will use the standard reduction techniques due to Fong and Reynolds. We review them here in terms of splendid and endo-permutation source Morita equivalences, which is less standard in the literature. These techniques work irrespective of the characteristic, so in this subsection p is an arbitrary prime.

Theorem 2.4 (Fong–Reynolds, see [KKW04, 1.5. Theorem])**.** *Let G be a finite group, let $N \trianglelefteq G$, let $\mathbf{b} \in \mathrm{Bl}(kN)$ and let $T := T_G(\mathbf{b})$ be the stabilizer of $\mathbf{b}$ in G. Then, block induction induces a bijection*

$$\mathrm{Bl}(kT \mid \mathbf{b}) \longrightarrow \mathrm{Bl}(kG \mid \mathbf{b}), \mathbf{A} \mapsto \mathbf{A}^G.$$

Moreover, if $\mathbf{B} \in \mathrm{Bl}(kG \mid \mathbf{b})$ and $\widetilde{\mathbf{B}} \in \mathrm{Bl}(kT \mid \mathbf{b})$ is the preimage of $\mathbf{B}$ under the latter bijection, then the pair of bimodules $(e_{\mathbf{B}} \cdot kG \cdot e_{\widetilde{\mathbf{B}}}, e_{\widetilde{\mathbf{B}}} \cdot kG \cdot e_{\mathbf{B}})$ realizes a splendid Morita equivalence between $\mathbf{B}$ and $\widetilde{\mathbf{B}}$.

Theorem 2.5 (Fong's Second Reduction Theorem, see [Lin18b, Theorem 6.8.13] and [Sal08, Proposition 3.3])**.** *Let G be a finite group and let $N \trianglelefteq G$ be a normal subgroup. Let $\mathbf{B} \in \mathrm{Bl}(kG)$ be a block covering a G-stable block $\mathbf{c} \in \mathrm{Bl}(kN)$ and set $S := kNe_{\mathbf{c}}$. If D is a defect group of $\mathbf{B}$ and $D \cap N = 1$, then $\mathbf{c}$ has defect zero and there is an isomorphism of interior D-algebras*

$$kGe_{\mathbf{c}} \cong kNe_{\mathbf{c}} \otimes_k k_\alpha[G/N]$$

where $\alpha \in \mathrm{H}^2(G/N, k^\times)$. Moreover, the following assertions hold.

(a) *There exists a central extension* $1 \longrightarrow Z \longrightarrow H \longrightarrow G/N \longrightarrow 1$*, where* Z *is the cyclic group of* $|G/N|_{p'}$*-th roots of unity in* k*, and there exists an idempotent* $e \in kZ$ *such that* $k_\alpha[G/N] \cong kHe$*.*

(b) *The block* $\mathbf{B}$ *is Morita equivalent to a block* $\widehat{\mathbf{B}}$ *of* kH *having a defect group* $\widehat{D}$ *isomorphic to* D*. After identifying these defect groups, the Morita equivalence is induced by a bimodule with vertex* ΔD *and an endo-permutation source.*

(c) *If* α *is trivial, then* H *can be replaced by* G/N *itself. If* G/N *is perfect, then* H *can be replaced by* $[H,H]$ *and* Z *by* $Z([H,H])$*.*

Detailed proofs of the main claim, of assertion (a) and of the second claim of assertion (c) can be found in [Sal08, §3 and §4]. The first claim of assertion (c) follows from [Lin18a, Proposition 1.2.6]. Assertion (b), and in particular the more precise information on the vertex and source of the bimodule inducing the Morita equivalence, will be crucial in our subsequent arguments. We record this information in the following remark. Its proof, together with a proof of Assertion (b), is given in Appendix A.

Remark 2.6. Since $\mathbf{c}$ has defect zero, the block algebra $S := kNe_{\mathbf{c}}$ is a matrix algebra. Thus, $S \cong \mathrm{End}_k(V)$ where V is, up to isomorphism, the unique simple $\mathbf{c}$-module. The module V has a unique extension, up to isomorphism, to a simple $k[ND]$-module, which we also denote by V. This extension has vertex D, and $\mathrm{Res}_D^{ND}(V)$ is a capped endo-permutation kD-module. Moreover, viewed as a $k[\Delta D]$-module via the identification $D \cong \Delta D$, the cap of $\mathrm{Res}_D^{ND}(V)$ is a source of the bimodule inducing the Morita equivalence constructed in the proof of Assertion (b).

2.7 Splendid stable equivalences of Morita type and trivial source modules. The following lemma generalizes parts of [KMN11, Lemma A.3].

Lemma 2.7. *Let* G *and* H *be two subgroups of a finite group* L*. Assume* G *and* H *have a common Sylow* p*-subgroup* P *and the same fusion system on* P*, i.e.* $\mathcal{F}_P(G) = \mathcal{F}_P(H)$*. Let* $\mathbf{A} \in \mathrm{Bl}(kG)$ *and let* $\mathbf{B} \in \mathrm{Bl}(kH)$ *be blocks with defect group* P*. Assume* M *(resp.* M'*) is an indecomposable finitely generated* $(\mathbf{A},\mathbf{B})$*-bimodule (resp. a* $(\mathbf{B},\mathbf{A})$*-bimodule), which, viewed as a* $k[G\times H]$*-module (resp. as a* $k[H\times G]$*-module), is a* p*-permutation module with vertex* ΔP*, and assume that the pair* (M,M') *induces a stable equivalence of Morita type between* $\mathbf{A}-\mathrm{mod}$ *and* $\mathbf{B}-\mathrm{mod}$*.*

Assume, furthermore, that X *is a non-projective indecomposable* $\mathbf{B}$*-module. Then, the following assertions hold.*

(a) *There exists a non-projective indecomposable* $\mathbf{A}$*-module* Y*, unique up to isomorphism, such that* $M \otimes_{\mathbf{B}} X = Y \oplus (\mathrm{proj})$ *as* $\mathbf{A}$*-modules.*

(b) *If* X *is a trivial source module, then so is* Y*. Moreover,* X *and* Y *have a vertex in common.*

Proof. Assertion (a) is well-known: it follows from [Lin18a, Proposition 4.13.2]. We prove (b). As X is non-projective, we may choose a vertex Q of X such that $1 \neq Q \leq P$ and we have

$$X \,\big|\, \mathrm{Ind}_Q^H(k) = kH \otimes_{kQ} k\,.$$

Now, by the assumption, we have $M \mid \mathrm{Ind}_{\Delta P}^{G\times H}(k)$, which is $kG \otimes_{kP} kH$ as (kG,kH)-bimodule. Therefore,

$$Y \,\big|\, (M \otimes_{kH} X) \,\big|\, kG \otimes_{kP} kH \otimes_{kH} kH \otimes_{kQ} k \,\big|\, \mathrm{Ind}_P^G\left(\mathrm{Res}_P^H(\mathrm{Ind}_Q^H(k))\right)$$

and the Mackey Formula yields

$$Y \,\Big|\, \bigoplus_{g\in[Q\backslash G/P]} \mathrm{Ind}_{{}^gQ\cap P}^{G}(k)\,.$$

Hence, Y is a trivial source module, and it has a vertex R such that $R \leq {}^gQ \cap P$ for some $g \in G$. It follows that ${}^{g^{-1}}R \leq Q \cap {}^{g^{-1}}P \leq Q \leq P$. Since $\mathcal{F}_P(G) = \mathcal{F}_P(H)$, there exists $h \in H$ such that ${}^{g^{-1}}R = {}^hR$, and so ${}^hR$ is also a vertex of Y and $R \leq_H Q$. Since the pair (M, M') induces a stable equivalence of Morita type, the same argument with M' yields $Q \leq_G R$. It follows that $R =_H Q$ and $Q =_G R$. Assertion (b) follows. □

3. Blocks with dihedral defect groups: preliminary results

To start with, we state two results, which do not require the defect of the blocks considered to be maximal. First, we restate [Mac22, Lemma 4.1] in terms of splendid Morita equivalences.

Lemma 3.1. *Let G be a finite group and let $N \trianglelefteq G$ be a normal subgroup such that $|G : N|$ is odd. If $\mathbf{b} \in \mathrm{Bl}(kN)$ is a block with dihedral defect groups of order 2^n ($n \geq 3$) and $\mathbf{B} \in \mathrm{Bl}(kG \mid \mathbf{b})$, then the $(\mathbf{B}, \mathbf{b})$-bimodule $e_{\mathbf{B}}kGe_{\mathbf{b}}$ and its dual realize a splendid Morita equivalence between $\mathbf{B}$ and $\mathbf{b}$. (The blocks $\mathbf{b}$ and $\mathbf{B}$ are* isomorphic via restriction *in the terminology of* [HK99, Theorem 4.1 and Definition 4.3]*.)*

Proof. First, observe that [Mac22, Lemma 4.1] proves that restriction from G to N induces a height-preserving bijection between $\mathrm{Irr}(\mathbf{B})$ and $\mathrm{Irr}(\mathbf{b})$. Thus, by [Bro90, 0.1 Théorème and 0.2 Théorème] the $(\mathbf{B}, \mathbf{b})$-bimodule $e_{\mathbf{B}}kGe_{\mathbf{b}}$ and its dual realize a Morita equivalence between $\mathbf{B}$ and $\mathbf{b}$. This equivalence is a splendid Morita equivalence, since $e_{\mathbf{B}}kGe_{\mathbf{b}}$ is a p-permutation bimodule. □

Lemma 3.2. *Let G be a finite soluble group. Then, any nilpotent block $\mathbf{B} \in \mathrm{Bl}(kG)$ with a defect group D isomorphic to a dihedral 2-group of order 2^n ($n \geq 3$) is splendidly Morita equivalent to kD_{2^n}.*

Proof. Let $\mathbf{B} \in \mathrm{Bl}(kG)$ be a nilpotent block and let A be a source algebra of $\mathbf{B}$. By Puig's characterisation of the source algebras of nilpotent blocks [Pui88] (see [Lin18b, Theorem 8.11.5]), there exists an endo-permutation kD-module V such that $A \cong \mathrm{End}_k(V) \otimes_k kD$ as interior D-algebras. Moreover, D is a vertex of V and V is a source of the unique simple $\mathbf{B}$-module. Thus, V is the trivial kD-module by Lemma 2.3(b) and the claim follows. □

4. Blocks of maximal defect: reduction to $G/O_{2'}(G)$

First, we use Fong's reductions to reduce the classification of blocks of maximal defect with dihedral defect groups up to splendid Morita equivalence to cyclic p'-central extensions of groups G with dihedral Sylow 2-subgroups and $O_{2'}(G) = 1$.

Lemma 4.1. *Let G be a finite group with dihedral Sylow 2-subgroups of order 2^n, where $n \geq 3$. Then any block $\mathbf{B} \in \mathrm{Bl}^{\max}(kG)$ is splendidly Morita equivalent to a block $\widehat{\mathbf{B}} \in \mathrm{Bl}_{\max}(k\widehat{L})$, where L is a finite group with $O_{2'}(L) = 1$ and dihedral Sylow 2-subgroups of order 2^n, and $\widehat{L}$ is a central extension of L by a cyclic group of order $|L|_{2'}$. Moreover, if $H^2(L, k^\times) = 1$, then $\widehat{L}$ may be replaced by L itself.*

Proof. We argue by induction on $|G|$. If $|G| = 8$, then $G \cong D_8$ and $\mathbf{B} = kG$. Taking $L = \widehat{L} = G$ and $\widehat{\mathbf{B}} = \mathbf{B}$ proves the assertion in this case. Now assume that $|G| > 8$ and that the assertion holds for every group satisfying the hypotheses whose order is strictly smaller than $|G|$. Set $N := O_{2'}(G)$, let $D \in \mathrm{Syl}_2(G)$, let $\mathbf{B} \in \mathrm{Bl}^{\max}(kG)$. Let $\mathbf{c} \in \mathrm{Bl}(kN)$ be a block covered by $\mathbf{B}$ and let T denote its stabilizer in G. Suppose first that $T < G$. By the Fong–Reynolds Theorem (Theorem 2.4) the block $\mathbf{B}$ is splendidly Morita equivalent to a block $\widetilde{\mathbf{B}} \in \mathrm{Bl}(kT)$. Moreover, $\widetilde{\mathbf{B}}$ has a defect group $D' \leq T$ which is G-conjugate to D. Since $\mathbf{B}$ has maximal defect, D' is a dihedral Sylow 2-subgroup of T of order 2^n, and $\widetilde{\mathbf{B}}$ has maximal defect. The assertion follows by applying the induction hypothesis to T and $\widetilde{\mathbf{B}}$, and using the transitivity of splendid Morita equivalences.

We may therefore assume that $T = G$, so that $\mathbf{c}$ is G-stable. Set $L := G/N$. Then $O_{2'}(L) = 1$. Since $D \cap N = 1$, the subgroup $DN/N \in \mathrm{Syl}_2(L)$ is isomorphic to D. Then, Theorem 2.5 (Fong's 2nd reduction) yields existence of a Morita equivalence between $\mathbf{B}$ and a block $\widehat{\mathbf{B}} \in \mathrm{Bl}_{\max}(k\widehat{L})$, where $\widehat{L}$ is a central extension of L by a cyclic group of order $|L|_{2'}$. Moreover, after identifying the defect groups, this Morita equivalence is induced by a bimodule with vertex ΔD and an endo-permutation source. This source is constructed as described in Remark 2.6: it can be identified with the cap of the endo-permutation kD-module $\mathrm{Res}_D^{ND}(V)$, say V_0, and we obtain that $V_0 \cong k$ by Lemma 2.3(b) as it is a source of a simple module for the 2-nilpotent (hence soluble) group ND and D is dihedral. This proves that the above Morita equivalence is a splendid Morita equivalence. The last claim is immediate from Theorem 2.5(c). □

Next, as a consequence of Theorem 3.1, we can describe all maximal defect 2-blocks with dihedral defect groups of the finite groups G with $O_{2'}(G) = 1$ up to splendid Morita equivalence.

Proposition 4.2. *Let G be a finite group with dihedral Sylow 2-subgroups of order 2^n ($n \geq 3$). Write $q = r^f$ for a power of a prime number r with f a positive integer. If $O_{2'}(G) = 1$, then any block $\mathbf{B} \in \mathrm{Bl}^{\max}(kG)$ is splendidly Morita equivalent to $\mathbf{B}_0(kG)$. Moreover, the following assertions hold.*

(a) *If $G = D_{2^n}$ with $n \geq 3$, then $\mathrm{Bl}^{\max}(kG) = \{\mathbf{B}_0(kG)\}$.*

(b) *If $n = 3$ and $G = \mathfrak{A}_7$, then $\mathrm{Bl}^{\max}(kG) = \{\mathbf{B}_0(kG)\}$.*

(c) *If $G = \mathrm{PSL}_2(q) \rtimes C_d$ with $(q-1)_2 = 2^n$ or $(q+1)_2 = 2^n$ and d is an odd integer such that $d \mid f$, then $|\mathrm{Bl}^{\max}(kG)| = d$ and any $\mathbf{B} \in \mathrm{Bl}^{\max}(kG)$ is splendidly Morita equivalent to $\mathbf{B}_0(k\mathrm{PSL}_2(q))$.*

(d) *If $G = \mathrm{PGL}_2(q) \rtimes C_d$ with $(q-1)_2 = 2^{n-1}$ or $(q+1)_2 = 2^{n-1}$ and d is an odd integer such that $d \mid f$, then $|\mathrm{Bl}^{\max}(kG)| = d$ and any $\mathbf{B} \in \mathrm{Bl}^{\max}(kG)$ is splendidly Morita equivalent to $\mathbf{B}_0(k\mathrm{PGL}_2(q))$.*

In cases (c) *and* (d)*, any choice of q satisfying the indicated condition yields a representative of the corresponding splendid Morita equivalence class.*

Proof. First, it follows from Subsection 2.4 that G is one of the groups listed in (a) to (d). Let $D \in \mathrm{Syl}_2(G)$. The number of blocks of maximal defect is easily calculated as follows. First, as D is dihedral of order at least 8, its automorphism group is a 2-group, so

$$N_G(D) = DC_G(D) = D \times O_{2'}(N_G(D)) \text{ and } C_G(D) = Z(D) \times O_{2'}(N_G(D)).$$

Then [NT12, Lemma 2.1] shows (as a corollary of Brauer's 1st Main Theorem) that $|\mathrm{Bl}^{\max}(kG)|$ is the number of $N_G(D)$-orbits on the conjugacy classes of elements in $O_{2'}(N_G(D))$.

Now, Case (a) is clear. Indeed, as $G = D$ is a 2-group, we have $kG = \mathbf{B}_0(kG)$. Case (b) is also clear as D is self-normalising in $G = \mathfrak{A}_7$, and so the principal block is the unique block of maximal defect by the above.

Thus, we can assume that G is one of the groups in (c) or (d). Set $G_0 := \mathrm{PSL}_2(q)$ in Case (c) and $G_0 := \mathrm{PGL}_2(q)$ in Case (d). Then $D \in \mathrm{Syl}_2(G_0)$ and we have $N_{G_0}(D) = D$. See [Bon11, Theorem 1.4.3(d)] for $\mathrm{PSL}_2(q)$. For $G_0 = \mathrm{PGL}_2(q)$ this easily follows from the fact that we may choose D to be the Sylow 2-subgroup of the normalizer of the split (resp. non-split) torus T when $q \equiv 1 \pmod 4$ (resp. $q \equiv 3 \pmod 4$) and $N_{G_0}(T) \cong D_{2(q-1)}$ (resp. $N_{G_0}(T) \cong D_{2(q+1)}$). Any element of $N_{G_0}(D)$ must then normalize the unique cyclic subgroup of D of index 2 and it follows that $N_{G_0}(D) \leq N_{G_0}(T)$ and $N_{G_0}(D) = D$. It follows that the principal block is the unique block of maximal defect of kG_0. If $d > 1$, then $N_G(D) = D \times C_d$ and we obtain that $|\operatorname{Bl}^{\max}(kG)| = d$ by the above counting argument. Moreover, any block $\mathbf{B} \in \operatorname{Bl}^{\max}(kG)$ covers a block $\mathbf{b} \in \operatorname{Bl}^{\max}(kG_0) = \{\mathbf{B}_0(kG_0)\}$ (as a defect group of $\mathbf{b}$ is G-conjugate to $D \cap G_0 = D$). By Lemma 3.1 any such block is splendidly Morita equivalent to $\mathbf{B}_0(kG_0)$. In particular, all the elements of $\operatorname{Bl}^{\max}(kG)$ are splendidly Morita equivalent to $\mathbf{B}_0(kG)$. Finally, in cases (c) and (d), the fact that the corresponding splendid Morita equivalence class does not depend on q is given by [KL19, Theorem 1.1]. The claim follows. □

5. The faithful blocks of maximal defect of $3.\mathfrak{A}_7$

In this section and in the next section, we prove that any block of maximal defect arising from the non-split central extensions originating from the non-trivial Schur multipliers in Lemma 2.2 is again splendidly Morita equivalent to a principal block. Our arguments require the known character and block theory of these groups, for which we use the notation of [CCN$^+$85] and [WTP$^+$]. For the purpose of our calculations in this section, we assume that $(K, \mathcal{O}, k)$ is a 2-modular system which is large enough for $3.\mathfrak{A}_7$ and all of its subgroups (this includes the groups $3.\mathfrak{A}_6$ and $\mathrm{PSL}_2(7) \cong \mathrm{GL}_3(2)$, which appear in Section 6) and where k is our previously fixed algebraically closed field of characteristic 2.

The triple cover $3.\mathfrak{A}_7$ of $\mathfrak{A}_7$ possesses two faithful 2-blocks of maximal defect, denoted Block 3 and Block 4 in [WTP$^+$, A7mod2], which are contragredient to each other. In the following, we prove that these blocks are splendidly Morita equivalent to $k\mathfrak{S}_4$. Recall here that $\mathfrak{S}_4 \cong \mathrm{PGL}_2(3)$. It is clear from Erdmann's work [Erd90] on tame blocks together with [Mac22, Theorem 2.1] that these blocks are Morita equivalent, because they have the same decomposition matrix. Erdmann's approach, classifying basic algebras, however, does not provide us with a bimodule realising the equivalence. We construct such a bimodule using the Brauer correspondence.

5.1 Context and notation. We first introduce the notation necessary to solve the problem.

Notation 5.1. Throughout this subsection, we use the following notation. First,

(1) $G := 3.\mathfrak{A}_7$ denotes the triple cover of $\mathfrak{A}_7$;
(2) $Z := Z(G) =: \langle z \rangle \cong C_3$;
(3) $P \in \mathrm{Syl}_2(G)$ is a fixed Sylow 2-subgroup, so $P \cong D_8$ and $N_G(P) = Z \times P \cong C_3 \times D_8$;
(4) $Q \lneq P$ denotes the unique subgroup such that $Q \cong C_2 \times C_2$ and $L := N_G(Q) = Z \times H$ with $H \cong \mathfrak{S}_4$, so $L \cong C_3 \times \mathfrak{S}_4$;

(5) $Q_2 \leq P$ denotes the second subgroup of P such that $Q_2 \cong C_2 \times C_2$, and its normaliser is $N_G(Q_2) \cong ((C_2 \times C_2) : ((C_3 \times C_3) : C_3)) : C_2$.

Thus, we have the following chain of subgroups:

$$Q \leq P \leq N_G(P) = Z \times P \leq L = Z \times H \leq G\,.$$

Next, we consider the following blocks:

(6) $\mathbf{A}$:= Block 3 of kG according to the notation of [WTP$^+$, A7mod2],
$\mathbf{A}^*$:= Block 4 of kG according to the notation of [WTP$^+$, A7mod2];

(7) $\mathbf{B}_0(kH) = kH$, the principal block of $H \cong \mathfrak{S}_4$ (all partitions of 4 have the same 2-core, namely the empty set, so it is clear that kH has just one block);

(8) $\mathbf{c}$ and $\mathbf{c}^*$, the Brauer correspondents in $L = N_G(Q)$ of $\mathbf{A}$ and $\mathbf{A}^*$ respectively.

Moreover, write:

(9) $\mathrm{Irr}(Z) := \{\zeta_0 := 1_{C_3}, \zeta_1, \zeta_2\}$ and for each $0 \leq i \leq 2$ let $e_i := e_{\zeta_i} \in kZ$ be the block idempotent associated to ζ_i. (See [NT89, Chapter 3, Theorem 6.22].) Therefore, $kZ = kZe_0 \oplus kZe_1 \oplus kZe_2$ is the block decomposition of kZ.

Clearly, the Sylow 2-subgroup P is a defect group of the blocks $\mathbf{A}$, $\mathbf{A}^*$, $\mathbf{B}_0(kH)$, $\mathbf{c}$ and $\mathbf{c}^*$, whereas the blocks of kZ are of defect zero.
The simple kH-modules and the simple $\mathbf{A}$-modules are as follows.

(10) We can read from [JK09, Appendix I.E] and/or the Character Table Library package `CTblLib` in GAP4 [GAP26] that the block $\mathbf{B}_0(kH) = kH$ has, up to isomorphism, two simple modules; the trivial module and a simple module of dimension 2, which are both liftable. Hence, we set $\mathrm{Irr}_k(\mathbf{B}_0(kH)) =: \{k_H, 2_H\}$.

(11) By [WTP$^+$, A7mod2, Block 3] the blocks $\mathbf{A}$ and $\mathbf{A}^*$ both have two simple modules: one of dimension 15 and one of dimension 6. Again, identifying these with their dimensions, we set $\mathrm{Irr}_k(\mathbf{A}) := \{15, 6\}$ and $\mathrm{Irr}_k(\mathbf{A}^*) := \{15^*, 6^*\}$. The 2-decomposition matrix of $\mathbf{A}$ and the values at the conjugacy classes of 2-elements are:

[CCN$^+$85, p.10]	[WTP$^+$]	15	6
χ_{17}	χ_{6_2}	0	1
χ_{18}	χ_{15_2}	1	0
χ_{19}	χ_{15_4}	1	0
χ_{20}	χ_{21_2}	1	1
χ_{21}	χ_{21_4}	1	1

	$1A$	$2A$	$4A$
χ_{17}	6	2	0
χ_{18}	15	-1	-1
χ_{19}	15	3	1
χ_{20}	21	1	-1
χ_{21}	21	-3	1

The block $\mathbf{A}$, seen as a left $k[G \times G]$-module, is indecomposable with vertex ΔP and trivial source. (See e.g. [NT89, Chap. 5, Theorem 10.8].) Moreover, by the definitions we have $N_{G\times G}(\Delta P) \leq N_G(P) \times N_G(P)$, so $N_{G\times G}(\Delta P) \leq G \times L$ and $N_{G\times G}(\Delta P) \leq L \times G$. Therefore, we may consider the Green correspondence $\mathfrak{f} := \mathfrak{f}(G \times G, \Delta P, L \times G)$ (following the notation of [NT89, Chap. 4, §4]). Thus, we may set

(12) $M := \mathfrak{f}(\mathbf{A})$, i.e. the unique indecomposable direct summand of $\mathrm{Res}^{G\times G}_{L\times G}(\mathbf{A})$ with vertex ΔP, which is a $(\mathbf{c}, \mathbf{A})$-bimodule.

5.2 A stable equivalence of Morita type. We first prove that the $(\mathbf{c}, \mathbf{A})$-bimodule M (and its dual) realise a stable equivalence of Morita type between $\mathbf{A}$ and $\mathbf{c}$ using Linckelmann's glueing method [Lin01, Theorem 3.1]. To this end, we collect information about the structure of the

centralizers of the 2-subgroups below and in the next remark, which can be found at [LMF26, Groups/Abstract/7560.g] or using GAP4 [GAP26].

Notation 5.2.

(13) We have $C_L(P) = Z \times Z(P) = C_G(P)$ and, under the identification

$$k[Z \times Z(P)] \cong kZ \otimes_k kZ(P)\,,$$

we have

$$kC_L(P)\,\mathrm{Br}_P(e_{\mathbf{c}}) = k[Z \times Z(P)](e_1 \otimes 1_{kZ(P)}) = kC_G(P)\,\mathrm{Br}_P(e_{\mathbf{A}})\,.$$

We identify e_1 with $e_1 \otimes 1_{kZ(P)}$. Hence,

$$f_P := \mathrm{Br}_P(e_{\mathbf{c}}) = e_1 = \mathrm{Br}_P(e_{\mathbf{A}}) =: e_P\,.$$

In particular, (P, f_P) and (P, e_P) are maximal Brauer pairs of $\mathbf{c}$ and $\mathbf{A}$, respectively.

(14) For any $R \leq P$, let f_R (resp. e_R) denote the unique block idempotent of $kC_L(R)$ (resp. $kC_G(R)$) such that

$$(R, f_R) \leq_L (P, f_P) \qquad \text{and} \qquad (R, e_R) \leq_G (P, e_P)\,.$$

The fusion systems of $\mathbf{c}$ and $\mathbf{A}$ with respect to the maximal Brauer pairs (P, f_P) and (P, e_P) are isomorphic, as both blocks have the same number of isomorphism classes of simple modules. (This follows from [AKO11, Part IV, Section 5.7] together with [Bra74]; see also [Sam14, Theorem 8.1(2)].) We identify them and denote the resulting fusion system on P by $\mathcal{F}$. The group P has two $\mathcal{F}$-conjugacy classes of involutions and

$$\mathrm{Aut}_{\mathcal{F}}(Q) \cong \mathfrak{S}_3 \quad \text{and} \quad \mathrm{Aut}_{\mathcal{F}}(Q_2) \cong C_2\,.$$

Remark 5.3. We record some local information that will be used repeatedly in the proof of Proposition 5.5 and which can be read off from [LMF26, Groups/Abstract/7560.g].

(a) Let $R \leq Q_2$ be a subgroup of order 2 such that $R \neq Z(P)$. Then, $C_G(R) = N \rtimes \hat{P}$ for some $\hat{P} \in \mathrm{Syl}_2(G) \setminus \{P\}$ containing Q_2, where $N := O_3(C_G(R)) \cong C_3 \times C_3$. As $C_G(R)$ contains $C_G(Q_2)$, we obtain that

$$C_G(Q_2) = C_{C_G(R)}(Q_2) = C_{N \rtimes \hat{P}}(Q_2) = C_N(Q_2) \rtimes C_{\hat{P}}(Q_2) = C_N(Q_2) \rtimes Q_2\,.$$

Since $|C_G(Q_2)| = 36$, we get that $C_N(Q_2) = N$, and hence $C_G(Q_2) = N \times Q_2$.

Next, as $Z(P) \leq Q_2$, we have $N \leq C_G(Q_2) \leq C_G(Z(P))$ and clearly $P \leq C_G(Z(P))$. Moreover, $Q_2 \trianglelefteq P$, so P normalizes $C_G(Q_2)$ and therefore also its characteristic subgroup $N = O_3(C_G(Q_2))$. Thus, $N \rtimes P \leq C_G(Z(P))$. Since $|C_G(Z(P))| = 72 = |N \rtimes P|$, we conclude that $C_G(Z(P)) = N \rtimes P$.

(b) Next, we record the corresponding local block idempotents. For every non-trivial subgroup $S \leq P$, one has $C_L(S) = Z \times C_P(S)$. Since $C_P(S)$ is a 2-group, under the natural identification $kC_L(S) \cong kZ \otimes_k kC_P(S)$ we have $f_S = e_1 \otimes 1_{kC_P(S)}$. As before, we identify e_1 with $e_1 \otimes 1_{kC_P(S)}$ and simply write $f_S = e_1$.

Since $C_G(Q_2) = N \times Q_2$ and Q_2 is a 2-group, there is a unique $\varepsilon \in \mathrm{Bl}(kN)$ such that $e_{Q_2} = \varepsilon \otimes 1_{kQ_2}$. Identifying ε with $\varepsilon \otimes 1_{kQ_2}$ as above, we simply write $e_{Q_2} = \varepsilon$. Since e_{Q_2} lies over the block e_1 of kZ, we have $\varepsilon e_1 = \varepsilon$.

Now, the block $e_{Z(P)}$ of $kC_G(Z(P))$ covers the same block ε of kN. We have $C_G(Z(P)) = N \rtimes P$, and $e_{Z(P)}$ has defect group $C_P(Z(P)) = P$. It follows that ε is P-stable, and hence $e_{Z(P)} = \varepsilon$.

Finally, let $R \leq Q_2$ be a subgroup of order 2 such that $R \neq Z(P)$. The block e_R of $kC_G(R)$ also covers ε. Recall from part (a) that $C_G(R) = N \rtimes \hat{P}$ for some $\hat{P} \in \mathrm{Syl}_2(G)$ containing Q_2. Since e_R has defect group $C_P(R) = Q_2$, the inertia group of ε in $\hat{P}$ is Q_2. Thus ε is not $\hat{P}$-stable and, for any $x \in \hat{P} \setminus Q_2$, we have $e_R = \mathrm{Tr}_{Q_2}^{\hat{P}} \varepsilon = \varepsilon + {}^x\varepsilon$.

Remark 5.4. Notice that for every $R \leq P$, we have

$$C_L(R) \times C_G(R) = C_{L\times G}(\Delta R) \leq N_{L\times G}(\Delta R) \leq N_L(R) \times N_G(R)\,.$$

If $R \cong C_2$, then any element normalising R also centralises R, thus in this case

$$C_{L\times G}(\Delta R) = C_L(R) \times C_G(R) = N_L(R) \times N_G(R) = N_{L\times G}(\Delta R)\,.$$

It follows that if Y is a $k[L \times G]$-module, its Brauer construction $Y[\Delta R]$ can be considered as a $k[C_L(R) \times C_G(R)]$-module via restriction, hence as a $(kC_L(R), kC_G(R))$-bimodule. We use this remark repeatedly in the proof of Proposition 5.5 below without further reference.

Proposition 5.5. *The $(\mathbf{c}, \mathbf{A})$-bimodule M defined in Notation 5.1(12) induces a stable equivalence of Morita type between the blocks $\mathbf{A}$ and $\mathbf{c}$.*

Proof. We prove the proposition using the glueing method given by [Lin01, Theorem 3.1]. Since the blocks $\mathbf{c}$ and $\mathbf{A}$ have isomorphic fusion systems, it suffices to show that for any nontrivial subgroup $R \leq P$, the bimodule $f_R M[\Delta R]e_R$ induces a splendid Morita equivalence between $kC_L(R)f_R$ and $kC_G(R)e_R$.

To this end, consider the $(\mathbf{c}, \mathbf{A})$-bimodule $\widetilde{M} := e_{\mathbf{c}} kG e_{\mathbf{A}}$. We have $\widetilde{M} \mid \mathrm{Res}^{G\times G}_{L\times G}(kGe_{\mathbf{A}})$. Moreover, the Brauer construction

$$\widetilde{M}[\Delta P] \cong \mathrm{Br}_P(e_{\mathbf{c}})kC_G(P)\,\mathrm{Br}_P(e_{\mathbf{A}}) = k[Z \times Z(P)]e_1$$

is non-zero. It follows that $\widetilde{M}$ has an indecomposable direct summand with vertex ΔP. The uniqueness of the Green correspondent implies that $M \mid \widetilde{M}$. Moreover, since M is the unique indecomposable summand of $\widetilde{M}$ with a vertex ΔP, the bimodule

$$f_P M[\Delta P]e_P = \mathrm{Br}_P(e_{\mathbf{c}})M[\Delta P]\,\mathrm{Br}_P(e_{\mathbf{A}}) = \mathrm{Br}_P(e_{\mathbf{c}})\widetilde{M}[\Delta P]\,\mathrm{Br}_P(e_{\mathbf{A}})$$

is non-zero. It follows that for any $R \leq P$, the bimodule $f_R M[\Delta R]e_R$ is a direct summand of $f_R\widetilde{M}[\Delta R]e_R$. Moreover, this summand is non-zero. Indeed, by [BP25, Proposition 5.3(c)], the pair $(\Delta P, f_P \otimes e_P^*)$ is a maximal M-Brauer pair. By [BP25, Remark 10.2(c)], we have

$$(\Delta R, f_R \otimes e_R^*) \leq (\Delta P, f_P \otimes e_P^*)\,.$$

Hence by [BP25, Proposition 5.3(b)], the pair $(\Delta R, f_R \otimes e_R^*)$ is an M-Brauer pair as well. Thus, $f_R M[\Delta R]e_R$ is non-zero. We now verify the required local splendid Morita equivalences case by case.

Case (i). Suppose that R is one of P, Q or the cyclic subgroup of order 4. Then in each of these cases, we have

$$C_L(R) = Z \times Z(R) = C_G(R)\,.$$

Indeed, $L = N_G(Q)$ implies that $C_G(Q) = C_L(Q) = Z \times Q$, and $N_G(P) = Z \times P$ implies that $C_G(P) = Z \times Z(P) = C_L(P)$. Moreover, if $R \cong C_4$, then $C_{\mathfrak{A}_7}(\overline{R}) = \overline{R}$ implies that $C_G(R) = Z \times R = C_L(R)$. Now, we have

$$kC_L(R)f_R = k[Z \times Z(R)]e_1 = kC_G(R)e_R$$

and hence the bimodule

$$f_R\widetilde{M}[\Delta R]e_R = e_1 k[Z \times Z(R)]e_1 = k[Z \times Z(R)]e_1$$

induces a splendid Morita equivalence between $kC_L(R)f_R$ and $kC_G(R)e_R$. In particular, the bimodule $f_R\widetilde{M}[\Delta R]e_R$ is indecomposable and thus $f_RM[\Delta R]e_R = f_R\widetilde{M}[\Delta R]e_R$.

Case (ii). Suppose that $R = Q_2$. By Remark 5.3,

$$C_L(R) = Z \times R, \qquad C_G(R) = N \times R, \qquad f_R = e_1, \qquad e_R = \varepsilon.$$

Therefore, the bimodule

$$f_R\widetilde{M}[\Delta R]e_R = e_1k[NR]\varepsilon$$

induces a splendid Morita equivalence between $kC_L(R)f_R = k[ZR]e_1$ and $kC_G(R)e_R = k[NR]\varepsilon$. As in Case (i), it follows that the bimodule $f_R\widetilde{M}[\Delta R]e_R$ is indecomposable. Thus we have $f_RM[\Delta R]e_R = f_R\widetilde{M}[\Delta R]e_R$.

Case (iii). Suppose that $R = Z(P)$. By Remark 5.3,

$$C_L(R) = Z \times P, \qquad C_G(R) = N \rtimes P, \qquad f_R = e_1, \qquad e_R = \varepsilon.$$

Therefore,

$$f_R\widetilde{M}[\Delta R]e_R = e_1k[N \rtimes P]\varepsilon$$

induces a splendid Morita equivalence between $k[Z \times P]e_1$ and $k[N \rtimes P]\varepsilon$. Hence it is indecomposable, and

$$f_RM[\Delta R]e_R = f_R\widetilde{M}[\Delta R]e_R.$$

Case (iv). Finally, suppose that $R \cong C_2$ and that R is not $\mathcal{F}$-conjugate to $Z(P)$. By Remark 5.3,

$$C_L(R) = Z \times Q_2\,, \quad C_G(R) = N \rtimes \hat{P}, \qquad f_R = e_1\,,$$

where $\hat{P} \in \mathrm{Syl}_2(G)$ contains Q_2, and $e_R = \varepsilon + {}^x\varepsilon$ for any $x \in \hat{P} \setminus Q_2$. Recall also that $e_{Q_2} = \varepsilon$ and $e_1\varepsilon = \varepsilon$.

By the Fong-Reynolds Theorem, the bimodule $\varepsilon k[C_G(R)]e_R$ induces a splendid Morita equivalence between $k[NQ_2]\varepsilon$ and $kC_G(R)e_R$. Moreover, the bimodule $e_1k[NQ_2]\varepsilon$ induces a splendid Morita equivalence between $k[ZQ_2]e_1$ and $k[NQ_2]\varepsilon$. Hence their composition

$$e_1k[NQ_2]\varepsilon \otimes_{k[NQ_2]\varepsilon} \varepsilon kC_G(R)e_R \cong \varepsilon kC_G(R)e_R$$

induces a splendid Morita equivalence between the blocks $k[ZQ_2]e_1 = kC_L(R)f_R$ and $k[N\hat{P}]e_R = kC_G(R)e_R$. In particular, the $(kC_L(R)f_R, kC_G(R)e_R)$-bimodule $\varepsilon kC_G(R)e_R$ is indecomposable. Note that

$$e_1e_R = e_1(\varepsilon + {}^x\varepsilon) = e_1\varepsilon + e_1{}^x\varepsilon = \varepsilon + {}^x\varepsilon = e_R$$

and hence

$$f_R\widetilde{M}[\Delta R]e_R = e_1kC_G(R)e_R = e_1e_RkC_G(R)e_R = \varepsilon kC_G(R)e_R \oplus {}^x\varepsilon kC_G(R)e_R$$

is a sum of two indecomposable modules. We need to show that $f_RM[\Delta R]e_R \cong \varepsilon kC_G(R)e_R$.

Set $Y := {}^x\varepsilon kC_G(R)e_R$ viewed as a $(kC_L(R), kC_G(R))$-bimodule. Since $f_RM[\Delta R]e_R$ is non-zero, it is enough to prove that Y is not a direct summand of $f_RM[\Delta R]e_R$. Suppose, for a contradiction, that it is. Note that $\varepsilon kC_G(R)e_R$ has vertex ΔQ_2 and ${}^x\varepsilon kC_G(R)e_R$ has vertex $\Delta_\varphi(Q_2) = \Delta(Q_2, \varphi, Q_2)$ where $\varphi = i_x\,|_{Q_2}\colon Q_2 \longrightarrow Q_2$.

Since $R \cong C_2$, we have $N_{L\times G}(\Delta R) = C_L(R) \times C_G(R)$, and by [Lin18a, Proposition 5.8.7],

$$M[\Delta R] \mid \mathrm{Res}^{L\times G}_{C_L(R)\times C_G(R)}(M).$$

Hence

$$Y \mid \mathrm{Res}^{L\times G}_{C_L(R)\times C_G(R)}(M).$$

The Brauer construction $Y[\Delta_\varphi(Q_2)]$ is naturally a module for $N_{C_L(R)\times C_G(R)}(\Delta_\varphi(Q_2))$, via inflation. Restricting to the centralizer, and using

$$C_{C_L(R)\times C_G(R)}(\Delta_\varphi(Q_2)) = C_L(Q_2) \times C_G(Q_2),$$

we see that $f_{Q_2} \otimes e^*_{Q_2}$ acts naturally on $Y[\Delta_\varphi(Q_2)]$. We claim that $(f_{Q_2} \otimes e^*_{Q_2})Y[\Delta_\varphi(Q_2)] \neq 0$. In fact, since ${}^x\varepsilon \otimes e^*_R$ is fixed by $C_L(R)\times C_G(R)$, by [BP25, Lemma 3.7], we have an isomorphism

$$Y[\Delta_\varphi(Q_2)] \cong \mathrm{Br}_{\Delta_\varphi(Q_2)}({}^x\varepsilon \otimes e^*_R)kC_G(R)[\Delta_\varphi(Q_2)]$$

of $kN_{C_L(R)\times C_G(R)}(\Delta_\varphi(Q_2))$-modules. Note that

$$N_{C_L(R)\times C_G(R)}(\Delta_\varphi(Q_2)) = C_L(R) \times C_G(Q_2).$$

Now both ${}^x\varepsilon$ and e_R are in kN, and Q_2 centralizes N. Hence $\mathrm{Br}_{\Delta_\varphi(Q_2)}({}^x\varepsilon \otimes e^*_R) = {}^x\varepsilon \otimes e^*_R$. Moreover, the $\Delta_\varphi(Q_2)$-fixed points of $C_G(R)$ are precisely $xC_G(Q_2)$. Indeed, for $g \in C_G(R)$ and $u \in Q_2$, one has $(\varphi(u), u)g = g$ if and only if $xux^{-1}gu^{-1} = g$. This holds for all $u \in Q_2$ if and only if $x^{-1}g \in C_G(Q_2)$. Thus, $kC_G(R)[\Delta_\varphi(Q_2)] \cong k[xC_G(Q_2)]$. Therefore,

$$Y[\Delta_\varphi(Q_2)] \cong {}^x\varepsilon k[xC_G(Q_2)]e_R = x\varepsilon kC_G(Q_2)e_R = x\varepsilon kC_G(Q_2)$$

where the last equality follows from $\varepsilon e_R = \varepsilon$. Finally, since $f_{Q_2} = e_1$, $e_{Q_2} = \varepsilon$, and $e_1\varepsilon = \varepsilon$, we obtain

$$(f_{Q_2} \otimes e^*_{Q_2})Y[\Delta_\varphi(Q_2)] \cong x\varepsilon kC_G(Q_2) \neq 0.$$

Since Y is a direct summand of $\mathrm{Res}^{L\times G}_{C_L(R)\times C_G(R)}(M)$, it follows that

$$(f_{Q_2} \otimes e^*_{Q_2})M[\Delta_\varphi(Q_2)] \neq 0.$$

Moreover, by [BP25, Remark 10.2(a)], $\big(\Delta_\varphi(Q_2), f_{Q_2} \otimes e^*_{Q_2}\big)$ is a Brauer pair of $L \times G$, and so by definition, it is an M-Brauer pair. By [BP25, Proposition 5.3(c)], this M-Brauer pair is contained in a maximal M-Brauer pair. Since $(\Delta P, f_P \otimes e^*_P)$ is a maximal M-Brauer pair and all maximal M-Brauer pairs are $(L \times G)$-conjugate, there exist $l \in L$ and $g \in G$ such that

$$\big(\Delta_\varphi(Q_2), f_{Q_2} \otimes e^*_{Q_2}\big) \leq {}^{(l,g)}\big(\Delta P, f_P \otimes e^*_P\big).$$

It follows that there exists a subgroup $U \leq P$ such that

$$\Delta_\varphi(Q_2) = {}^{(l,g)}\Delta U \quad \text{and} \quad f_{Q_2} \otimes e^*_{Q_2} = {}^{(l,g)}(f_U \otimes e^*_U) = {}^l f_U \otimes {}^g e^*_U\,.$$

This means that

$${}^lU = Q_2 = {}^gU, \qquad f_{Q_2} = {}^l f_U, \qquad e_{Q_2} = {}^g e_U \qquad \text{and} \qquad \varphi = i_l \circ i_g^{-1} : Q_2 \longrightarrow Q_2\,.$$

Therefore, the maps $i_l|_U : U \longrightarrow Q_2$ and $i_g|_U : U \longrightarrow Q_2$ are isomorphisms in the fusion systems of $\mathbf{c}$ and $\mathbf{A}$, respectively. Under our identification of these two fusion systems, both are isomorphisms in $\mathcal{F}$. Since Q and Q_2 are not $\mathcal{F}$-conjugate, it follows that $U = Q_2$. Hence both $i_l|_{Q_2}$ and $i_g|_{Q_2}$ are in $\mathrm{Aut}_\mathcal{F}(Q_2)$, and therefore $\varphi \in \mathrm{Aut}_\mathcal{F}(Q_2)$. But $\mathrm{Aut}_\mathcal{F}(Q_2) = \mathrm{Aut}_P(Q_2)$.

Now φ is non-trivial since $\varphi = i_x|_{Q_2}$ for $x \in \hat{P} \setminus Q_2$. On the other hand, $\hat{P} \leq C_G(R)$, and hence x centralizes R. Thus $\varphi = i_x|_{Q_2}$ fixes R. This gives a contradiction. Indeed, the unique non-trivial element of $\mathrm{Aut}_P(Q_2)$ fixes $Z(P)$ and interchanges the other two subgroups of order 2 in Q_2. Since $R \neq Z(P)$, no non-trivial element of $\mathrm{Aut}_P(Q_2) = \mathrm{Aut}_\mathcal{F}(Q_2)$ can fix R.

Therefore, Y is not a direct summand of $f_R M[\Delta R]e_R$ and we conclude that

$$f_R M[\Delta R]e_R \cong \varepsilon kC_G(R)e_R.$$

Hence $f_R M[\Delta R]e_R$ induces the required splendid Morita equivalence. □

5.3 The splendid Morita equivalences. It now remains to prove that the stable equivalence of Proposition 5.5 is a splendid Morita equivalence. We achieve this using a method due to Linckelmann (see [Lin96b, Theorem 2.1(ii) and (iii)]), which consists in proving that the functor $M \otimes_{\mathbf{A}} -$ maps the simple $\mathbf{A}$-modules to the simple $\mathbf{c}$-modules. Thus, we first need to collect information on the simple modules of the blocks involved.

Lemma 5.6. *The following assertions hold.*

(a) *We have* $|\operatorname{Bl}(kL)| = 3$. *Furthermore,*

$$\mathbf{B}_0(kL) = kZe_0 \otimes_k \mathbf{B}_0(kH), \quad \mathbf{c} = kZe_1 \otimes_k \mathbf{B}_0(kH), \text{ and } \mathbf{c}^* = kZe_2 \otimes_k \mathbf{B}_0(kH)$$

(permuting the labelling of e_1 *and* e_2 *if necessary) and all three blocks are splendidly Morita equivalent to* $\mathbf{B}_0(kH)$.

(b) *The block* $\mathbf{c}$ *contains precisely two pairwise non-isomorphic simple modules, one of dimension* 1 *and one of dimension* 2. *We identify these modules with their dimensions and write* $\operatorname{Irr}_k(\mathbf{c}) =: \{1, 2\}$.

(c) *The simple* $\mathbf{c}$*-module* 1 *is the unique trivial source* $\mathbf{c}$*-module with vertex* P. *The simple* $\mathbf{c}$*-module* 2 *is a trivial source module with vertex* Q.

Proof. (a) Since Z is a $2'$-group and $|\operatorname{Irr}(Z)| = 3$, we have $|\operatorname{Bl}(kZ)| = 3$ and each block of kZ is of the form kZe_i for some $i \in \{0, 1, 2\}$ (as described in Notation 5.1(9)). By Notation 5.1(7), $\operatorname{Bl}(kH) = \{\mathbf{B}_0(kH)\}$. Thus, it is clear from Subsection 2.3 that $|\operatorname{Bl}(kL)| = 3$ and any block of kL has the given form. The last claim follows from Lemma 2.1.

(b) By (a), any simple $\mathbf{c}$-module is of the form $S_1 \otimes_k T$, where S_1 is the unique simple kZe_1-module and T is a simple $\mathbf{B}_0(kH)$-module. Now, $\dim_k(S_1) = 1$ and by Notation 5.1(10) we have $\operatorname{Irr}_k(\mathbf{B}_0(kH)) = \{k_H, 2_H\}$. Hence $1 := S_1 \otimes_k k_H$ is a simple module of dimension 1 and $2 := S_1 \otimes_k 2_H$ is a simple module of dimension 2. The claim follows.

(c) It is clear that k_H is a trivial source kH-module with vertex P. Next, as $H \cong \mathfrak{S}_4$, $Q \trianglelefteq H$ and $\mathfrak{S}_4/Q \cong \mathfrak{S}_3$, any trivial source kH-module with vertex Q is the inflation from H/Q to H of a PIM (see e.g. [Las23, Theorem 4.6(c)]). Moreover, we can read from [JK09, Appendix I.E] and/or the Character Table Library package `CTblLib` in GAP4 [GAP26] that the 2-decomposition matrix of $k\mathfrak{S}_3$ is

partition notation	degree	$k_{\mathfrak{S}_3}$	$2_{\mathfrak{S}_3}$
$\chi_{(3)} = 1_{\mathfrak{S}_3}$	1	1	0
$\chi_{(1^3)} =: \operatorname{sgn}_{\mathfrak{S}_3}$ (sign character)	1	1	0
$\chi_{(2,1)}$	2	0	1

where $2_{\mathfrak{S}_3}$ denotes a simple $k\mathfrak{S}_3$-module of dimension 2. It follows that $2_H = \operatorname{Inf}_{H/Q}^H(2_{\mathfrak{S}_3})$ and is a trivial source kH-module with vertex Q. In addition, the number of isomorphism classes of trivial source kH-modules with vertex P is given by the number of $2'$-conjugacy classes of $N_H(P)/P = P/P \cong \{1\}$, i.e. one. (See e.g. [Las23, Theorem 4.6(c)].)

Now, as $\mathbf{c}$ is splendidly Morita equivalent to $\mathbf{B}_0(kH)$, it follows that the simple $\mathbf{c}$-modules 1 and 2 are trivial source $\mathbf{c}$-modules with vertex P and Q respectively. Also, since $\mathbf{B}_0(kH)$ contains, up to isomorphism, a unique trivial source module with vertex P, so does $\mathbf{c}$, as claimed.

□

Lemma 5.7. *Let* f *denote the Green correspondence with respect to* (G, L, P) *from* G *to* $L = N_G(Q)$. *Then, the following assertions hold.*

(a) *The simple* $\mathbf{A}$*-module* 15 *is a trivial source module with vertex* P *and* $\chi_{\widehat{15}} = \chi_{19}$.

(b) *The simple* $\mathbf{A}$*-module* 6 *is a trivial source module with vertex* Q *and* $\chi_{\widehat{6}} = \chi_{17}$.

(c) *We have* $f(15) = 1$ *and* $f(6) = 2$.

Proof. We prove the three claims together. First, we can read from the ATLAS [CCN$^+$85, p.10] that G has a maximal subgroup $F \cong C_3 \times \mathfrak{S}_5$. Now, letting k_{ζ_1} denote the simple kC_3-module corresponding to the non-trivial character $\zeta_1 \in \mathrm{Irr}(C_3)$, the 2-permutation kG-module $X := \mathrm{Ind}_F^G(k_{\zeta_1} \otimes_k k_{\mathfrak{S}_5})$ affords the character

$$\chi_{\widehat{X}} = \mathrm{Ind}_F^G(\zeta_1 \otimes 1_{\mathfrak{S}_5}) = \chi_{17} + \chi_{19}$$

(where the decomposition into irreducibles is easily obtained using GAP 4 [GAP26]). Hence, we read from the 2-decomposition matrix of $\mathbf{A}$ given in Notation 5.1(11) that X is an $\mathbf{A}$-module and has two composition factors, each with multiplicity one, namely 6 and 15. Moreover, [Lan83, II Theorem 12.4] yields $\dim_k(\mathrm{End}_{kG}(X)) = \langle \chi_{\widehat{X}}, \chi_{\widehat{X}} \rangle_G = 2$, and so X cannot be uniserial of length 2, proving that $X = 15 \oplus 6$.

It follows that both the simple $\mathbf{A}$-modules 15 and 6 are trivial source kG-modules with $\chi_{\widehat{6}} = \chi_{17}$ and $\chi_{\widehat{15}} = \chi_{19}$. Clearly, P is a vertex of 15 since its k-dimension is odd. Moreover, $\dim_k(6) = 2{\cdot}3$ implies that 6 has vertices of order 4 (see [Oku81, Lemma 2.2]), and as $\chi_{\widehat{6}}(2A) = 2$ and $\chi_{\widehat{6}}(4A) = 0$ (see Notation 5.1(11)), by [Lan83, II Lemma 12.6(ii)+(iii)], the vertices of 6 cannot be cyclic of order 4, so they are Klein-four subgroups. Then, to prove that Q is a vertex of 6, we prove that Q_2 cannot be a vertex of 6. Indeed, restricting $\chi_{\widehat{6}}$ to $N_G(Q_2)$ using GAP yields

$$\mathrm{Res}^G_{N_G(Q_2)}(\chi_{\widehat{6}}) = \eta_1 + \eta_2$$

with $\eta_1, \eta_2 \in \mathrm{Irr}(N_G(Q_2))$ both of degree 3 and taking values 1 and -1 on the unique conjugacy class of elements of order 4. It follows that $\mathrm{Res}^G_{N_G(Q_2)}(6)$ is indecomposable (indeed, if it decomposed, one of the summands would afford a character with a negative value at some 2-element, which is impossible for a 2-permutation module by [Lan83, II Lemma 12.6(ii)]). This means that $\mathrm{Res}^G_{N_G(Q_2)}(6)$ is not simple, because it has at least two composition factors. Consequently, Q_2 cannot be a vertex of 6 since then its Green correspondent in $N_G(Q_2)$ would have to be simple by Okuyama's Theorem [Oku81, Theorem 2.3].

Next, $f(15)$ and $f(6)$ lie in $\mathbf{c}$ as the Green correspondence and the Brauer correspondence commute. Since $\mathbf{c}$ contains a unique trivial source module with vertex P by Lemma 5.6(c), clearly $f(15) = 1$. Finally, $f(6)$ is also simple by Okuyama's Theorem [Oku81, Theorem 2.3]. Thus, the only possibility is $f(6) = 2$. □

Lemma 5.8. *Let* M *be the* $(\mathbf{c}, \mathbf{A})$*-bimodule defined in Notation 5.1*(12)*. Then, the following assertions hold:*

(a) $M \otimes_{\mathbf{A}} 15 \cong 1$ *and* $M \otimes_{\mathbf{A}} 6 \cong 2$ *as* $\mathbf{c}$*-modules;*

(b) M *and its dual realise a splendid Morita equivalence between* $\mathbf{A}$ *and* $\mathbf{c}$.

Proof. By Proposition 5.5, we already know that M realises a stable equivalence of Morita type between $\mathbf{A}$ and $\mathbf{c}$. Therefore, we may use [Lin96b, Theorem 2.1(ii) and (iii)].

(a) First, by [Lin96b, Theorem 2.1(ii)], the $\mathbf{c}$-modules $M \otimes_{\mathbf{A}} 15$ and $M \otimes_{\mathbf{A}} 6$ are indecomposable and non-projective, since 15 and 6 are non-projective simple $\mathbf{A}$-modules. Moreover, by Lemma 5.7, the simple modules 15 and 6 are trivial source modules with vertex P and Q respectively. Thus, we know from [KMN11, Lemma A.3(ii) and (iii)] that $M \otimes_{\mathbf{A}} 15$ is a

trivial source module with vertex P and $M \otimes_{\mathbf{A}} 6$ is a trivial source module with vertex Q. Now, $M = \mathfrak{f}(\mathbf{A}) | \operatorname{Res}^{G\times G}_{L\times G}(\mathbf{A})$ implies that

$$M \otimes_{\mathbf{A}} S \mid \operatorname{Res}^{G\times G}_{L\times G}(\mathbf{A}) \otimes_{\mathbf{A}} S = \operatorname{Res}^{G}_{L}(S)$$

for each $S \in \operatorname{Irr}_k(\mathbf{A}) = \{15, 6\}$. By the properties of the Green correspondence, we know that

$$\operatorname{Res}^{G}_{L}(15) = f(15) \oplus X \qquad \text{and} \qquad \operatorname{Res}^{G}_{L}(6) = f(6) \oplus Y\,,$$

where X (respectively Y) is a direct sum of indecomposable kL-modules with a vertex which is not L-conjugate to P (respectively not L-conjugate to Q). Thus, it follows from Lemma 5.7(c) that $M \otimes_{\mathbf{A}} 15 = f(15) = 1$ and $M \otimes_{\mathbf{A}} 6 = f(6) = 2$.

(b) By (a), for any simple $\mathbf{A}$-module S, the $\mathbf{c}$-module $M \otimes_{\mathbf{A}} S$ is simple. Therefore, M realises a Morita equivalence between $\mathbf{A}$ and $\mathbf{c}$ by [Lin96b, Theorem 2.1(iii)]. Since by definition M is a 2-permutation bimodule, this equivalence is splendid, as required.

□

Theorem 5.9. *The two faithful blocks $\mathbf{A}, \mathbf{A}^* \in \operatorname{Bl}^{\max}(k[3.\mathfrak{A}_7])$ are splendidly Morita equivalent to the principal block $\mathbf{B}_0(k\mathfrak{S}_4) = k\mathfrak{S}_4$ of the symmetric group $\mathfrak{S}_4$.*

Proof. By Lemma 5.8(b) the block $\mathbf{A}$ is splendidly Morita equivalent to its Brauer correspondent $\mathbf{c}$ in $L = N_G(Q)$. As the Brauer correspondence and the Green correspondence commute with duality, analogous arguments show that $\mathbf{A}^*$ is splendidly Morita equivalent to its Brauer correspondent $\mathbf{c}^*$. Furthermore, by Lemma 5.6, both the blocks $\mathbf{c}$ and $\mathbf{c}^*$ are splendidly Morita equivalent to $\mathbf{B}_0(kH) = kH$, where $H \cong \mathfrak{S}_4$. The claim follows. □

6. The faithful blocks of maximal defect of $3.\mathfrak{A}_6$

The group $\mathrm{PSL}_2(9)$ is isomorphic to $\mathfrak{A}_6$, and its triple cover $3.\mathfrak{A}_6$ possesses two faithful 2-blocks of maximal defect, denoted Block 4 and Block 5 in [WTP$^+$, A6mod2], which are contragredient to each other. Below, we prove that these blocks are splendidly Morita equivalent to the principal block of $\mathrm{PSL}_2(7)$. Again, it is clear from [Erd90] and [Mac22, Theorem 2.1] that these blocks are Morita equivalent, because they have the same decomposition matrix, but Erdmann's approach does not provide us with a bimodule realising the equivalence. We construct such a bimodule.

We note, however, that contrary to the previous case, the Brauer correspondence cannot be used, as it is easily checked using GAP4 [GAP26] and [Mac22, Theorem 2.1] that none of the subgroups L of $3.\mathfrak{A}_6$ with $|L|_2 = 8$ has blocks with the same decomposition matrix. Moreover, not all simple modules involved are trivial source modules.

6.1 Context and notation. As in Section 5, we first introduce the notation necessary to solve the problem carefully.

Notation 6.1. Throughout this subsection, we use the following notation. First,

(1) $G := 3.\mathfrak{A}_7$ as in Section 5;
(2) $H := \mathrm{PSL}_2(7) \cong \mathrm{GL}_3(2)$;
(3) $F := 3.\mathfrak{A}_6$ denotes the triple cover of $\mathfrak{A}_6$;
(4) $Z := Z(F) = Z(G) =: \langle z\rangle \cong C_3$.

We may, and do, assume that F and H are subgroups of G.

(5) We set $J := H \cap F \leq G$ and note that $J \cong \mathfrak{S}_4$.

(6) We fix a Sylow 2-subgroup $P \in \mathrm{Syl}_2(G)$ such that $P \in \mathrm{Syl}_2(H)$ and $P \in \mathrm{Syl}_2(F)$. We have $P \cong D_8$, $N_F(P) = Z \times P \cong C_3 \times D_8$ and $N_H(P) = P \cong D_8$.

(7) We let $Q, Q_2 \leq P$ denote the two Klein-four subgroups of P as in Notation 5.1. These are not F-conjugate. Notice, however, that in this case $N_F(Q) \cong C_3 \times \mathfrak{S}_4$ and also $N_F(Q_2) \cong C_3 \times \mathfrak{S}_4$ (where the C_3 factor is $Z(F)$) but $N_F(Q) \neq N_F(Q_2)$.

Thus, we have the following chains of subgroups:

$$Q < P < N_F(P) = N_G(P) < N_F(Q) = N_G(Q) < F < G$$

and

$$Q < P = N_H(P) < N_H(Q) < H < G\,.$$

Next, we consider the following blocks:

(8) $\mathbf{B} := \mathbf{B}_0(kH)$, the principal block of kH;

(9) $\mathbf{C}$:= Block 4 of kF according to the notation of [WTP$^+$, A6mod2],
$\mathbf{C}^*$:= Block 5 of kF according to the notation of [WTP$^+$, A6mod2].

Without loss of generality we may assume that $\mathbf{C}$ and $\mathbf{C}^*$ are the Brauer correspondents of the blocks $\mathbf{A}$ and $\mathbf{A}^*$ of $k[3.\mathfrak{A}_7]$, respectively. Moreover, as in Section 5, we write:

(10) $\mathrm{Irr}(Z) := \{\zeta_0 := 1_{C_3}, \zeta_1, \zeta_2\}$, $e_i := e_{\zeta_i} \in kZ$ for the block idempotent associated to ζ_i for each $0 \leq i \leq 2$, hence $kZ = kZe_0 \oplus kZe_1 \oplus kZe_2$ is the block decomposition of kZ.

We have that P is a defect group of $\mathbf{C}$, $\mathbf{C}^*$ and $\mathbf{B}$, whereas the blocks of kZ are of defect zero. The simple $\mathbf{C}$-modules and the simple $\mathbf{B}$-modules are as follows.

(11) By [WTP$^+$, L3(2)mod2] the principal block $\mathbf{B}$ of $k\,\mathrm{PSL}_2(7)$ has three simple modules: the trivial module k_H and two simple modules of dimension 3. Following [WTP$^+$, L3(2)mod2] we set $\mathrm{Irr}_k(\mathbf{B}) := \{k_H, \tilde{3}_1, \tilde{3}_2\}$. The 2-decomposition matrix of $\mathbf{B}$ and the values at the conjugacy classes of 2-elements are:

[CCN$^+$85, p.3]	[WTP$^+$]	$\tilde{3}_1$	$\tilde{3}_2$	k_H
χ_2	χ_{3_1}	1	0	0
χ_3	χ_{3_2}	0	1	0
χ_4	χ_{6_1}	1	1	0
χ_1	χ_{1_1}	0	0	1
χ_5	χ_{7_1}	1	1	1

	$1A$	$2A$	$4A$
χ_2	3	-1	1
χ_3	3	-1	1
χ_4	6	2	0
χ_1	1	1	1
χ_5	7	-1	-1

(12) By [WTP$^+$, A6mod2] the blocks $\mathbf{C}$ and $\mathbf{C}^*$ both have three simple modules: two of dimension 3 and one of dimension 9. Following [WTP$^+$, A6mod2] we set $\mathrm{Irr}_k(\mathbf{C}) := \{3_1, 3_3, 9_1\}$ and $\mathrm{Irr}_k(\mathbf{C}^*) := \{3_1^*, 3_3^*, 9_1^*\}$. The 2-decomposition matrix of $\mathbf{C}$ and the values at the conjugacy classes of 2-elements are:

[CCN$^+$85, p.4]	[WTP$^+$]	3_1	3_3	9_1
χ_{14}	χ_{3_1}	1	0	0
χ_{15}	χ_{3_3}	0	1	0
χ_{16}	χ_{6_1}	1	1	0
χ_{17}	χ_{9_2}	0	0	1
χ_{18}	χ_{15_1}	1	1	1

	$1A$	$2A$	$4A$
χ_{14}	3	-1	1
χ_{15}	3	-1	1
χ_{16}	6	2	0
χ_{17}	9	1	1
χ_{18}	15	-1	-1

6.2 A stable equivalence of Morita type. As in Subsection 5.2 we first prove a stable equivalence of Morita type between $\mathbf{C}$ and $\mathbf{B}$ using Linckelmann's glueing method [Lin01, Theorem 3.1]. For this purpose, we collect information about the fusion systems of $\mathbf{C}$ and $\mathbf{B}$ and about the structure of the centralisers of the 2-subgroups, which can be found at [LMF26, Groups/Abstract/1080.260 and Groups/Abstract/168.42] or using GAP4 [GAP26].

Notation 6.2.

(13) We have $C_H(P) = Z(P) \cong C_2$. It follows that $kC_H(P) \cong kC_2$ has just one block idempotent, hence $\mathrm{Br}_P^H(e_{\mathbf{B}}) = 1_{kC_H(P)} =: e_P$.
We have $C_F(P) = Z \times Z(P) \cong C_3 \times C_2$ and, under the identification

$$k[Z \times Z(P)] \cong kZ \otimes_k kZ(P)\,,$$

we have $kC_F(P)\,\mathrm{Br}_P^F(e_{\mathbf{C}}) = k[Z \times Z(P)](e_1 \otimes 1_{kZ(P)})$. (Up to permuting the labelling of e_1 and e_2 if necessary.) We identify e_1 with $e_1 \otimes 1_{kZ(P)}$ and we set

$$f_P := \mathrm{Br}_P^F(e_{\mathbf{C}}) = e_1\,.$$

In particular, (P, f_P) and (P, e_P) are maximal Brauer pairs of $\mathbf{C}$ and $\mathbf{B}$, respectively.

(14) For any $R \leq P$, let e_R (resp. f_R) denote the unique block idempotent of $kC_H(R)$ (resp. $kC_F(R)$) such that

$$(R, e_R) \leq_H (P, e_P) \qquad \text{and} \qquad (R, f_R) \leq_F (P, f_P)\,.$$

The fusion systems of $\mathbf{B}$ and $\mathbf{C}$ with respect to the maximal Brauer pairs (P, e_P) and (P, f_P) are isomorphic, as both blocks have the same number of isomorphism classes of simple modules. (This follows from [AKO11, Part IV, Section 5.7] together with [Bra74]; see also [Sam14, Theorem 8.1(2)].) We identify them and denote the resulting fusion system on P by $\mathcal{F}$. In this case, P has one $\mathcal{F}$-conjugacy class of involutions, namely the class of the central involution.

(15) If $R \leq P$ is one of P, Q, Q_2, $Z(P)$ or the cyclic group of order 4, then one has

$$C_H(R) = C_P(R) \qquad \text{and} \qquad C_F(R) = Z \times C_P(R)\,.$$

Since $C_P(R)$ is a 2-group, under the natural identification $kC_F(R) \cong kZ \otimes_k kC_P(R)$, we have $f_R = e_1 \otimes 1_{kC_P(R)}$. As before, identifying e_1 with $e_1 \otimes 1_{kC_P(R)}$, we simply write $f_R = e_1$.

Remark 6.3. Let $R \leq P$. Then

$$C_H(R) \times C_F(R) = C_{H\times F}(\Delta R) \leq N_{H\times F}(\Delta R) \leq N_H(R) \times N_F(R)\,.$$

The latter inclusion is an equality if and only if both $N_H(R)$ and $N_F(R)$ centralise R. In particular, if $R \cong C_2$, we have

$$C_{H\times F}(\Delta R) = C_H(R) \times C_F(R) = N_H(R) \times N_F(R) = N_{H\times F}(\Delta R)\,.$$

For any $R \leq P$ and any $k[H \times F]$-module Y, the Brauer construction $Y[\Delta R]$ is naturally a $kN_{H\times F}(\Delta R)$-module. By restriction to $C_{H\times F}(\Delta R) = C_H(R) \times C_F(R)$, we may therefore regard $Y[\Delta R]$ as a $(kC_H(R), kC_F(R))$-bimodule, with bimodule structure given by

$$h \cdot y \cdot f := (h, f^{-1})y$$

for every $h \in C_H(R)$, $f \in C_F(R)$ and $y \in Y[\Delta R]$.

Next we define a series of bimodules, which will help us prove the required Morita equivalence between $\mathbf{B}$ and $\mathbf{C}$.

Notation 6.4.

(16) Set $\widetilde{X} := \mathrm{Ind}_{\Delta P}^{H\times F}(k)$, and consider the $(\mathbf{B},\mathbf{C})$-bimodule $X := e_{\mathbf{B}}\widetilde{X}e_{\mathbf{C}}$

(17) Set $\widetilde{W} := \mathrm{Ind}_{\Delta J}^{H\times F}(k)$ and consider the $(\mathbf{B},\mathbf{C})$-bimodule $W := e_{\mathbf{B}}\widetilde{W}e_{\mathbf{C}}$.

Lemma 6.5. *Let R be one of P, Q, Q_2 or the cyclic group of order 4. Then*

$$\widetilde{W}[\Delta R] \cong k[N_{H\times F}(\Delta R)/\Delta N_J(R)]$$

as $kN_{H\times F}(\Delta R)$-modules, where the left action on the permutation module on the right-hand side is induced by left multiplication.

Proof. First note that $\widetilde{W}$ is isomorphic to the permutation $k[H\times F]$-module $k\Omega$, where $\Omega := (H\times F)/\Delta J$. Hence, by [Lin18a, Proposition 5.8.1], $\widetilde{W}[\Delta R] \cong k[\Omega^{\Delta R}]$ as $kN_{H\times F}(\Delta R)$-modules, where $\Omega^{\Delta R}$ denotes the set of ΔR-fixed points in Ω, and we identify each element of $\Omega^{\Delta R}$ with its image under the Brauer map.

A coset $(h,f)\Delta J$ is ΔR-fixed if and only if

$$h^{-1}rh = f^{-1}rf \in J \qquad \text{for every } r \in R\,.$$

For each of the subgroups R under consideration, every subgroup of J which is F-conjugate to R is J-conjugate to R. Hence, in this case, $h^{-1}Rh = f^{-1}Rf$ is J-conjugate to R. Choose $x \in J$ such that $x^{-1}h^{-1}Rhx = R$. Then

$$(h,f)\Delta J = (hx,fx)\Delta J \qquad \text{and} \qquad (hx,fx) \in N_{H\times F}(\Delta R)\,.$$

Conversely, every coset represented by an element of $N_{H\times F}(\Delta R)$ is ΔR-fixed. Thus $\Omega^{\Delta R}$ is the $N_{H\times F}(\Delta R)$-orbit of ΔJ. Since the stabilizer of the coset ΔJ in $N_{H\times F}(\Delta R)$ is

$$N_{H\times F}(\Delta R) \cap \Delta J = \Delta N_J(R)\,,$$

we obtain

$$\Omega^{\Delta R} \cong N_{H\times F}(\Delta R)/\Delta N_J(R)$$

as $N_{H\times F}(\Delta R)$-sets. Linearizing this isomorphism proves the result. □

Lemma 6.6. *The following assertions hold.*

(a) *We have $N_{H\times F}(\Delta P) = \Delta P\cdot(1\times(Z\times Z(P)))$ and the map*

$$N_{H\times F}(\Delta P)/\Delta P \longrightarrow Z\times Z(P), \qquad (a,b)\Delta P \mapsto ab^{-1}$$

is an isomorphism.

(b) *We have $\widetilde{W}[\Delta P] \cong k[Z\times Z(P)]$ and $W[\Delta P] \cong k[Z\times Z(P)]e_1$ as $kN_{H\times F}(\Delta P)$-modules. Here the action of $N_{H\times F}(\Delta P)$ on $Z\times Z(P)$ is the transport, via the isomorphism in part* (a)*, of the left multiplication action on $N_{H\times F}(\Delta P)/\Delta P$. More explicitly,*

$$(u,v)\cdot x = uxv^{-1}$$

for $(u,v) \in N_{H\times F}(\Delta P)$ and $x \in Z\times Z(P)$. In particular, $W[\Delta P]$ is indecomposable.

(c) *The bimodule X, seen as a $k[H\times F]$-module, has a unique indecomposable summand with vertex ΔP, say M. In particular, M is, up to isomorphism, the unique (kH,kF)-bimodule with vertex ΔP that can realise a splendid Morita equivalence between $\mathbf{B}$ and $\mathbf{C}$.*

(d) *The $k[H\times F]$-module W has a unique direct summand with vertex ΔP, and this summand is isomorphic to the $k[H\times F]$-module M in Part* (c).

Proof. (a) Since $N_F(P) = Z \times P$ and $N_H(P) = P$, intersecting $N_{G\times G}(\Delta P)$ with $H \times F$, [BP25, Proposition 2.4(a)] gives

$$N_{H\times F}(\Delta P) = \Delta P \cdot (\{1\} \times C_F(P)) = \Delta P \cdot (\{1\} \times (Z \times Z(P)))\,.$$

The second claim is immediate after noting that $\Delta P \cap (\{1\} \times (Z \times Z(P))) = 1$.

(b) Since $N_J(P) = P$, by Lemma 6.5, $\widetilde{W}[\Delta P] \cong k[N_{H\times F}(\Delta P)/\Delta P]$. Transporting the left multiplication action through the isomorphism in part (a) gives

$$\widetilde{W}[\Delta P] \cong k[Z \times Z(P)]$$

as $kN_{H\times F}(\Delta P)$-modules. Indeed, if $(u, v) \in N_{H\times F}(\Delta P)$ and $x \in Z \times Z(P)$, then the transported action is given by $(u, v) \cdot x = uxv^{-1}$.

By [BP25, Lemma 3.7],

$$W[\Delta P] \cong \mathrm{Br}_P(e_{\mathbf{B}})\widetilde{W}[\Delta P]\,\mathrm{Br}_P(e_{\mathbf{C}}) \cong k[Z \times Z(P)]e_1.$$

Finally, upon restriction to $Z(P) \times 1 \leq C_H(P) \times C_F(P)$, the latter module is isomorphic to the regular $kZ(P)$-module which is indecomposable. Hence $W[\Delta P]$ is indecomposable.

(c) Since $\widetilde{X}$ is the permutation $k[H \times F]$-module $k[(H \times F)/\Delta P]$, we have

$$\widetilde{X}[\Delta P] \cong \mathrm{Ind}_{\Delta P}^{N_{H\times F}(\Delta P)}(k) \cong k[N_{H\times F}(\Delta P)/\Delta P]$$

as $k[N_{H\times F}(\Delta P)]$-modules; see e.g. [Lin18a, Corollary 5.8.4]. Therefore, by part (a) and [BP25, Lemma 3.7], we have

$$X[\Delta P] \cong \mathrm{Br}_P(e_{\mathbf{B}})\widetilde{X}[\Delta P]\,\mathrm{Br}_P(e_{\mathbf{C}}) \cong k[Z \times Z(P)]e_1\,.$$

By part (b), the latter module is indecomposable and the claim follows from the Broué correspondence; see [Bro85, Theorem (3.2) and (3.4)].

(d) By part (b), W has a unique indecomposable summand, say M', with vertex ΔP. Hence

$$M' \mid \mathrm{Ind}_{\Delta P}^{H\times F}(k) = \widetilde{X}.$$

Moreover, since M' is a direct summand of $W = e_{\mathbf{B}}\widetilde{W}e_{\mathbf{C}}$, the block idempotents $e_{\mathbf{B}}$ and $e_{\mathbf{C}}$ act as the identity on M'. Thus M' is in fact a direct summand of $X = e_{\mathbf{B}}\widetilde{X}e_{\mathbf{C}}$. By part (c), $M' \cong M$, as required.

□

Notation 6.7.

(18) Let M denote the unique indecomposable direct summand of X with vertex ΔP given by Lemma 6.6(c), which we may see as a $(\mathbf{B}, \mathbf{C})$-bimodule.

Lemma 6.8. *For every subgroup $R \leq P$, one has $e_R M[\Delta R]f_R \neq 0$.*

Proof. By Lemma 6.6(b),

$$e_P W[\Delta P]f_P = W[\Delta P] \cong k[Z \times Z(P)]e_1 \neq 0\,.$$

By Lemma 6.6(d), M is, up to isomorphism, the unique direct summand of W with vertex ΔP. It follows that $e_P M[\Delta P]f_P \neq 0$. Thus, by [BP25, Proposition 5.3(c)], $(\Delta P, e_P \otimes f_P^*)$ is a maximal M-Brauer pair. By [BP25, Remark 10.2(c)], we have $(\Delta R, e_R \otimes f_R^*) \leq (\Delta P, e_P \otimes f_P^*)$. Hence, by [BP25, Proposition 5.3(b)], the pair $(\Delta R, e_R \otimes f_R^*)$ is an M-Brauer pair, as well. Thus, $e_R M[\Delta R]f_R$ is non-zero. □

Lemma 6.9. *Let $R = Q_2$. Then $e_R M[\Delta R] f_R \cong k[Z \times R]e_1$ as $(kC_H(R)e_R, kC_F(R)f_R)$-bimodules. In particular, $e_R M[\Delta R] f_R$ induces a splendid Morita equivalence between $kC_H(R)e_R$ and $kC_F(R)f_R$.*

Proof. First note that $N_H(R) \cong \mathfrak{S}_4$ and $N_F(R) \cong Z \times \mathfrak{S}_4$. Choose an involution $t \in P \setminus R$, so that $P = R \rtimes \langle t \rangle$. Choose complements S_H to R in $N_H(R)$ and S_F to $Z \times R$ in $N_F(R)$, both containing t. Note that both S_H and S_F are isomorphic to $\mathfrak{S}_3$. Let $\theta : S_H \longrightarrow S_F$ be an isomorphism such that s and $\theta(s)$ induce the same automorphism of R for every $s \in S_H$. In particular, $\theta(t) = t$. Then

$$N := N_{H\times F}(\Delta R) = \{(as, zb\theta(s)) \in N_H(R) \times N_F(R) \mid a, b \in R, z \in Z, s \in S_H\}\,.$$

Define

$$\Phi : N \longrightarrow Z \times N_H(R), \qquad (as, zb\theta(s)) \mapsto (z^{-1}, ab^{-1}s)\,.$$

A direct calculation shows that Φ is a surjective homomorphism with $\ker(\Phi) = \Delta R$ and $\Phi(\Delta P) = 1 \times \langle t \rangle$. Consider $(Z \times N_H(R)/(1 \times \langle t \rangle))$ as an N-set via Φ, that is, for $n \in N$ and $x \in Z \times N_H(R)$, we set

$$n \cdot x(1 \times \langle t \rangle) = \Phi(n)x(1 \times \langle t \rangle)\,.$$

Since $\ker(\Phi) = \Delta R \leq \Delta P$, we have $\Phi^{-1}(1 \times \langle t \rangle) = \Delta P$. Thus, Φ induces an isomorphism of N-sets

$$\overline{\Phi} : N/\Delta P \longrightarrow (Z \times N_H(R))/(1 \times \langle t \rangle), \qquad n\Delta P \mapsto \Phi(n)(1 \times \langle t \rangle)\,.$$

Using the identification $(Z \times N_H(R))/(1 \times \langle t \rangle) \cong Z \times (N_H(R)/\langle t \rangle)$, we obtain an isomorphism of N-sets

$$N/\Delta P \cong Z \times (N_H(R)/\langle t \rangle)\,.$$

Since $N_J(R) = P$, together with Lemma 6.5, this implies that we have isomorphisms

$$\widetilde{W}[\Delta R] \cong k[N/\Delta P] \cong k[Z \times (N_H(R)/\langle t \rangle)]$$

of kN-modules. Therefore,

$$\widetilde{W}[\Delta R] \cong kZ \otimes \mathrm{Ind}_{\langle t \rangle}^{N_H(R)}(k)$$

as kN-modules, where the N-action on the latter module is induced by Φ. More precisely, for every $n = (as, zb\theta(s)) \in N$, $z_0 \in Z$ and $x \in N_H(R)$,

$$n \cdot (z_0 \otimes (x \otimes 1)) = z^{-1}z_0 \otimes (ab^{-1}sx \otimes 1)\,.$$

We now decompose the $kN_H(R)$-module $\mathrm{Ind}_{\langle t \rangle}^{N_H(R)} k$. Since $\langle t \rangle \leq S_H \leq N_H(R)$, we get

$$\mathrm{Ind}_{\langle t \rangle}^{N_H(R)}(k) \cong \mathrm{Ind}_{S_H}^{N_H(R)}(\mathrm{Ind}_{\langle t \rangle}^{S_H}(k))\,.$$

But $\mathrm{Ind}_{\langle t \rangle}^{S_H}(k) \cong k \oplus E$ as kS_H-modules, where E is the two-dimensional simple projective kS_H-module. Hence $\mathrm{Ind}_{\langle t \rangle}^{N_H(R)}(k) \cong U_0 \oplus V_0$, where $U_0 = \mathrm{Ind}_{S_H}^{N_H(R)}(k)$ and $V_0 = \mathrm{Ind}_{S_H}^{N_H(R)}(E)$. The module U_0 is indecomposable with vertex $\langle t \rangle$. Indeed, $\mathrm{Res}_R^{N_H(R)}(U_0) \cong kR$ is indecomposable, and thus U_0 is also indecomposable. Also, k is a direct summand of $\mathrm{Ind}_{\langle t \rangle}^{S_H}(k)$, so the transitivity of induction yields $U_0 \mid \mathrm{Ind}_{\langle t \rangle}^{N_H(R)}(k)$. Thus, U_0 is relatively $\langle t \rangle$-projective. Since $\dim_k(U_0) = 4$ is not divisible by $|N_H(R)|_2 = 8$, U_0 is not projective. Thus, $\langle t \rangle$ is a vertex of U_0. Since induction preserves projectivity, V_0 is projective. Since kZe_1 is a one-dimensional projective as a kZ-module, the $k[Z \times N_H(R)]$-module $kZe_1 \otimes_k U_0$ is indecomposable with vertex $1 \times \langle t \rangle$. In addition, $kZe_1 \otimes_k V_0$ is projective.

Recall that $e_R = 1$ and $f_R = e_1$. Thus,

$$e_R\widetilde{W}[\Delta R]f_R \cong kZe_1 \otimes_k \mathrm{Ind}_{\langle t \rangle}^{N_H(R)}(k) \cong U \oplus V$$

where $U = kZe_1 \otimes_k U_0$ and $V = kZe_1 \otimes V_0$ are viewed as kN-modules through Φ. Note that the indecomposable kN-module U has vertex $\Phi^{-1}(1 \times \langle t \rangle) = \Delta P$ and every indecomposable direct summand of V has vertex $\ker(\Phi) = \Delta R$.

Since $N_{\Delta P}(\Delta R) = \Delta P$, ΔR is fully normalized in $\mathcal{F}_{\Delta P}(H \times F)$. Since also M is an indecomposable trivial source $k[H \times F]$-module with vertex ΔP, [KS23, Lemma 6.4] implies that every indecomposable direct summand of $M[\Delta R]$, as a kN-module, has a vertex properly containing ΔR. In particular, the same holds for every indecomposable direct summand of $e_R M[\Delta R] f_R$. By Lemma 6.6(d), M is isomorphic to a direct summand of W, hence $e_R M[\Delta R] f_R$ is isomorphic to a direct summand of $e_R \widetilde{W}[\Delta R] f_R$. By Lemma 6.8, this summand is non-zero. It follows from the preceding paragraph that $e_R M[\Delta R] f_R \cong U$ as kN-modules.

Now we restrict the kN-module $U = kZe_1 \otimes_k U_0$ to $C := C_{H\times F}(\Delta R) = C_H(R) \times C_F(R) = R \times (Z \times R)$. For $(a, zb) \in R \times (Z \times R)$, we have $\Phi(a, zb) = (z^{-1}, ab^{-1})$. Thus,

$$\mathrm{Res}^N_C(U) \cong kZe_1 \otimes_k \mathrm{Res}^{N_H(R)}_R(U_0) \cong kZe_1 \otimes kR$$

as kC-modules. Here, kC acts on the latter module via $(a, zb) \cdot (u \otimes y) = z^{-1}u \otimes ab^{-1}y$ for every $a, b \in R, z \in Z$, $u \in kZe_1$ and $y \in kR$. The standard k-algebra isomorphism

$$\psi : kZe_1 \otimes_k kR \longrightarrow k[Z \times R]e_1, \qquad u \otimes x \mapsto ux$$

is in fact an isomorphism of kC-modules where kC acts on $k[Z \times R]e_1$ by $(a, zb) \cdot w = aw(zb)^{-1}$ for every $a, b \in R$, $z \in Z$ and $w \in k[Z \times R]e_1$. Indeed, for $a, b \in R$, $z \in Z$, $u \in kZe_1$ and $y \in kR$, we have

$$\psi((a, zb)\cdot(u\otimes y)) = \psi(z^{-1}u\otimes ab^{-1}y) = z^{-1}uab^{-1}y = a(uy)(zb)^{-1} = (a, zb)\cdot(uy) = (a, zb)\cdot\psi(u\otimes y).$$

By Remark 6.3, this kC-module structure on $k[Z \times R]e_1$ corresponds to the usual $(kR, k[Z \times R])$-bimodule structure given by left and right multiplication. This proves both assertions. □

Proposition 6.10. *The $(\mathbf{B}, \mathbf{C})$-bimodule M defined in Notation 6.7(18) and its dual induce a stable equivalence of Morita type between the blocks $\mathbf{B}$ and $\mathbf{C}$.*

Proof. By [Lin01, Theorem 3.1] and the isomorphism of the fusion systems of $\mathbf{B}$ and $\mathbf{C}$, it suffices to show that the bimodule $e_R M[\Delta R] f_R$ induces a splendid Morita equivalence between $kC_H(R)e_R$ and $kC_F(R)f_R$ for representatives R of the non-trivial $\mathcal{F}$-conjugacy classes of subgroups of P. We may take $R \in \{P, Q, Q_2, Z(P), C_4\}$, where C_4 denotes the cyclic subgroup of P of order 4. By Lemma 6.8, all bimodules $e_R M[\Delta R] f_R$ are non-zero.

Case (i). Suppose that $R = P$. By Lemma 6.6(d), M is isomorphic to a direct summand of W. Hence $e_P M[\Delta P] f_P$ is isomorphic to a non-zero direct summand of $W[\Delta P]$. By Lemma 6.6(b), the latter module is indecomposable, and therefore $e_P M[\Delta P] f_P \cong W[\Delta P] \cong k[Z \times Z(P)]e_1$. By Remark 6.3, this is an isomorphism of $(kC_H(P)e_P, kC_F(P)f_P)$-bimodules. Thus the required splendid Morita equivalence follows.

Case (ii). Suppose that $R = Q$ or $R \cong C_4$. In both cases, $C_H(R) = R$, $C_F(R) = Z \times R$ and $N_{H\times F}(\Delta R) = \Delta N_J(R) \cdot (\{1\} \times (Z \times R))$. Indeed, if $(a, b) \in N_{H\times F}(\Delta R)$, then $a \in N_J(R)$ and $a^{-1}b \in C_F(R)$. The map

$$N_{H\times F}(\Delta R)/\Delta N_J(R) \longrightarrow Z \times R, \qquad (a, b)\Delta N_J(R) \mapsto ab^{-1},$$

is an isomorphism of $N_{H\times F}(\Delta R)$-sets, where the action on the latter set is given by $(u, v) \cdot x = uxv^{-1}$ for every $(u, v) \in N_{H\times F}(\Delta R)$ and $x \in Z \times R$. Thus Lemma 6.5 gives $\widetilde{W}[\Delta R] \cong k[Z \times R]$ as $k[N_{H\times F}(\Delta R)]$-modules. By [BP25, Lemma 3.7],

$$W[\Delta R] \cong \mathrm{Br}_R(e_{\mathbf{B}})\widetilde{W}[\Delta R]\,\mathrm{Br}_R(e_{\mathbf{C}}) \cong k[Z \times R]e_1$$

as $k[N_{H\times F}(\Delta R)]$-modules. The latter module is indecomposable. By Lemma 6.6(d) and Lemma 6.8, $e_R M[\Delta R] f_R$ is isomorphic to a non-zero direct summand of $e_R W[\Delta R] f_R$. Hence $e_R M[\Delta R] f_R \cong k[Z \times R] e_1$. By Remark 6.3, this is an isomorphism of $(kC_H(R)e_R, kC_F(R)f_R)$-bimodules. The latter bimodule induces the desired splendid Morita equivalence.

Case (iii). The case $R = Q_2$ follows from Lemma 6.9.

Case (iv). Finally, Suppose that $R = Z(P)$. Since $R \cong C_2$, we have $N_H(R) = C_H(R) = P$, $N_F(R) = C_F(R) = Z \times P$ and

$$N_{H\times F}(\Delta R) = C_H(R) \times C_F(R) = P \times (Z \times P)\,.$$

Thus, the $k[N_{H\times F}(\Delta R)]$-module $M[\Delta R]$ is already a $k[C_H(R)\times C_F(R)]$-module. Since $e_R = 1_{kP}$ and $f_R = e_1$, the $(kC_H(R), kC_F(R))$-bimodule $Y := e_R M[\Delta R] f_R$ is naturally a $(kP, k[Z\times P]e_1)$-bimodule. We first show that Y is indecomposable. So write

$$Y = Y_1 \oplus \cdots \oplus Y_t \qquad (t \in \mathbb{Z}_{>0})$$

as a direct sum of indecomposable $k[N_{H\times F}(\Delta R)]$-modules. Since $N_{\Delta P}(\Delta R) = \Delta P$, ΔR is fully normalized in $\mathcal{F}_{\Delta P}(H \times F)$. Thus by [KS23, Lemma 6.4], every indecomposable direct summand of the $k[N_{H\times F}(\Delta R)]$-module $M[\Delta R]$ has a vertex S satisfying

$$\Delta R \lhd S \leq_{N_{H\times F}(\Delta R)} \Delta P. \tag{1}$$

Since $Y \mid M[\Delta R]$, this conclusion holds in particular for Y_i for each $1 \leq i \leq t$, which can therefore only have vertices of order 4 or 8.

Now as $R = Z(P)$ is characteristic in P, $N_{H\times F}(\Delta P) \leq N_{H\times F}(\Delta R)$. Hence, the transitivity of the Brauer construction [Lin18a, Proposition 5.8.5] yields $(M[\Delta R])[\Delta P] \cong M[\Delta P]$ as $k[N_{H\times F}(\Delta P)]$-modules. By the structures of the centralisers described above, we obtain $\mathrm{Br}_P^{C_H(R)}(e_R) = e_P$ and $\mathrm{Br}_P^{C_F(R)}(f_R) = f_P$. Since e_R and f_R are P-stable, [BP25, Lemma 3.7] gives

$$Y[\Delta P] \cong \mathrm{Br}_P^{C_H(R)}(e_R)\,(M[\Delta R])\,[\Delta P]\,\mathrm{Br}_P^{C_F(R)}(f_R) \cong e_P M[\Delta P] f_P$$

as $(kC_H(P)e_P, kC_F(P)f_P)$-bimodules, which is non-zero and indecomposable by Case (i). On the other hand, the Brauer construction preserves direct sums, so $Y[\Delta P] = Y_1[\Delta P]\oplus\cdots\oplus Y_t[\Delta P]$. It follows that, after relabelling if necessary,

$$Y_1[\Delta P] \neq 0 \qquad \text{and} \qquad Y_i[\Delta P] = 0 \qquad \text{for every } 2 \leq i \leq t\,.$$

As Y_1 is a trivial source module, the non-vanishing of $Y_1[\Delta P]$ implies that ΔP is contained in a vertex of Y_1. By (1), such a vertex has order at most $|P|$. Hence ΔP itself is a vertex of Y_1. Now fix an index i such that $2 \leq i \leq t$. As Y_i is also a trivial source module, the vanishing of $Y_i[\Delta P]$ shows that Y_i cannot have a vertex which is $N_{H\times F}(\Delta R)$-conjugate to ΔP and so (1) yields that it has a vertex which is $N_{H\times F}(\Delta R)$-conjugate to ΔT, for some subgroup $T \leq P$ satisfying $R < T < P$. Therefore, $|T| = 4$ and we may assume that ΔT itself is a vertex of Y_i. Again, because Y_1 and Y_i are trivial source modules, we have $Y_1[\Delta T] \neq 0$ and $Y_i[\Delta T] \neq 0$, and it follows that $Y[\Delta T] = Y_1[\Delta T] \oplus \cdots \oplus Y_t[\Delta T]$ is decomposable. On the other hand, since $\Delta R \lhd \Delta T$, the transitivity of the Brauer construction gives $Y[\Delta T] \cong e_T M[\Delta T] f_T$ as $(kC_H(T)e_T, kC_F(T)f_T)$-bimodules, where the latter bimodule is non-zero and indecomposable by Cases (ii) and (iii), contradicting the decomposability of $Y[\Delta T]$. Therefore, $t = 1$ and Y is indecomposable.

It remains to identify Y. The coset $\Delta J \in (H \times F)/\Delta J$ is fixed by ΔR, and its stabilizer in $N_{H\times F}(\Delta R)$ is

$$N_{H\times F}(\Delta R) \cap \Delta J = \Delta C_J(R) = \Delta P.$$

Hence by [Lin18a, Proposition 5.8.1], the $N_{H\times F}(\Delta R)$-orbit of ΔJ yields a direct summand $k[N_{H\times F}(\Delta R)/\Delta P]$ of $\widetilde{W}[\Delta R]$ as $k[N_{H\times F}(\Delta R)]$-modules. Multiplying by $e_R = 1_{kP}$ and $f_R = e_1$, we obtain a direct summand $k[N_{H\times F}(\Delta R)/\Delta P]e_1$ of $W[\Delta R]$ as $k[N_{H\times F}(\Delta R)]$-modules.

The map

$$N_{H\times F}(\Delta R)/\Delta P \longrightarrow Z \times P, \qquad (a, zb)\Delta P \mapsto a(zb)^{-1}$$

is an isomorphism of $N_{H\times F}(\Delta R)$-sets, where $N_{H\times F}(\Delta R) = P \times (Z \times P)$ acts on $Z \times P$ by

$$(u, zv) \cdot x = ux(zv)^{-1}$$

for every $u, v \in P$, $z \in Z$ and $x \in Z \times P$. Therefore, $k[N_{H\times F}(\Delta R)/\Delta P]e_1 \cong k[Z \times P]e_1$ as $k[N_{H\times F}(\Delta R)]$-modules. The latter module is indecomposable, since its restriction to $P \times 1$ is isomorphic to the regular kP-module. Moreover, $(k[Z \times P]e_1)[\Delta P] \cong k[Z \times Z(P)]e_1 \neq 0$, since the set of ΔP-fixed points of $Z \times P$ is equal to $Z \times Z(P)$ and e_1 is P-stable.

On the other hand, as above, since $R = Z(P)$ is characteristic in P, the transitivity of the Brauer construction [Lin18a, Proposition 5.8.5] gives

$$(W[\Delta R])[\Delta P] \cong W[\Delta P],$$

as $k[N_{H\times F}(\Delta P)]$-modules. The latter module is non-zero and indecomposable by Lemma 6.6(b). Therefore, $W[\Delta R]$ has a unique indecomposable direct summand whose Brauer construction at ΔP is non-zero. By Lemma 6.6(d), Y is isomorphic to a direct summand of $W[\Delta R]$. We have already shown that Y is indecomposable and that $Y[\Delta P] \neq 0$. Hence, by the preceding paragraph,

$$Y \cong k[N_{H\times F}(\Delta R)/\Delta P]e_1 \cong k[Z \times P]e_1$$

as $k[N_{H\times F}(\Delta R)]$-modules. By Remark 6.3, this is an isomorphism of $(kP, k[Z\times P]e_1)$-bimodules. This bimodule induces the required splendid Morita equivalence. □

6.3 The splendid Morita equivalences. Again, it remains to prove that the stable equivalence of Proposition 6.10 is a splendid Morita equivalence, which we achieve using [Lin96b, Theorem 2.1(ii) and (iii)]. We start by collecting information on the simple and trivial source modules of the blocks involved.

Lemma 6.11.

(a) *The trivial kH-module k_H is the unique trivial source $\mathbf{B}$-module with vertex P and we have $\chi_{\widehat{k_H}} = \chi_1$.*

(b) *The block $\mathbf{B}$ contains precisely two pairwise non-isomorphic trivial source modules with vertex Q, namely $\mathrm{Sc}(H, Q)$ and a uniserial module $\widetilde{U} := U(\widetilde{3}_1, \widetilde{3}_2)$ (permuting the labelling of $\widetilde{3}_1$ and $\widetilde{3}_2$ if necessary) affording the character $\chi_{\widehat{\widetilde{U}}} = \chi_4$.*

(c) *Let $\mathbf{b}$ denote the Brauer correspondent of $\mathbf{B}$ in $N_H(Q) \cong \mathfrak{S}_4$ and let $f_{H,Q}$ denote the Green correspondence with respect to $(H, Q, N_H(Q))$ from H to $N_H(Q)$. Then, $f_{H,Q}(\widetilde{U}) = 2_{\mathfrak{S}_4}$, the unique two-dimensional $\mathbf{b}$-module. (See Notation 5.1(10).)*

Before we proceed to the proofs of Lemma 6.11 and the next lemma, we note that since $H \leq G$ and $F \leq G$, we may consider again the simple kG-module 15 (respectively the simple kG-module 6) defined in Notation 5.1(11), which is a trivial source module with vertex P (respectively Q) by Lemma 5.7. These allow us to calculate the relevant trivial source modules of kH and kF.

Proof. (a) It is immediate that k_H is a trivial source module with vertex P and belongs to $\mathbf{B}$. Moreover, the number of isomorphism classes of trivial source kH-modules with vertex P is given by the number of $2'$-conjugacy classes of $N_H(P)/P = P/P \cong \{1\}$, i.e. one. (See e.g. [Las23, Theorem 4.6(c)].) The claim follows.

(b) Similarly, the number of isomorphism classes of trivial source kH-modules with vertex Q equals the number of $2'$-conjugacy classes of $N_H(Q)/Q \cong \mathfrak{S}_4/Q \cong \mathfrak{S}_3$, which is two. Since kH has two blocks, namely $\mathbf{B} = \mathbf{B}_0(kH)$ and a block of defect zero, any trivial source kH-module with vertex Q belongs to $\mathbf{B}$. Certainly, $\mathrm{Sc}(H, Q)$ is such a module.

Next, by [Lan83, II, Lemma 12.6(ii)], $\mathrm{Res}_H^G(\chi_{\widehat{6}})$ takes non-negative integer values at all 2-elements. Therefore, it follows from Notation 6.1(11) that the only possibility is

$$\mathrm{Res}_H^G(\chi_{\widehat{6}}) = \chi_4\,.$$

Thus, $\mathrm{Res}_H^G(6)$ is a trivial source module with vertex Q. We then read from the decomposition matrix of $\mathbf{B}$ that $\mathrm{Res}_H^G(6)$ is uniserial of length 2 with composition factors $\widetilde{3}_1$ and $\widetilde{3}_2$. The claim follows.

(c) We have $N_H(Q) \cong \mathfrak{S}_4$ and $N_G(Q) \cong C_3 \times \mathfrak{S}_4$. Moreover, the block $\mathbf{b}$ is splendidly Morita equivalent to the block $\mathbf{c}$ of $N_G(Q)$ (defined in Notation 5.1(8)), see Lemma 5.6(a). Thus, if f denotes the Green correspondence with respect to $(G, Q, N_G(Q))$ from G to $N_G(Q)$, then Lemma 5.7(c) yields

$$f_{H,Q}(\widetilde{U}) = \mathrm{Res}^{N_G(Q)}_{N_H(Q)}(f(6)) = \mathrm{Res}^{N_G(Q)}_{N_H(Q)}(2) = 2_{\mathfrak{S}_4}\,.$$

□

Lemma 6.12.

(a) *The simple kF-module 9_1 is the unique trivial source $\mathbf{C}$-module with vertex P and we have $\chi_{\widehat{9_1}} = \chi_{17}$.*

(b) *The block $\mathbf{C}$ contains precisely two pairwise non-isomorphic trivial source modules with vertex Q, one of which is the uniserial module $U := U(3_1, 3_3)$ affording the character $\chi_{\widehat{U}} = \chi_{16}$ (permuting the labelling of 3_1 and 3_3 if necessary).*

(c) *Let $\mathbf{c}$ be the Brauer correspondent of $\mathbf{C}$ in $N_F(Q) = N_G(Q) \cong C_3 \times \mathfrak{S}_4$ and let $f_{F,Q}$ denote the Green correspondence with respect to $(F, Q, N_F(Q))$ from F to $N_F(Q)$. Then, $f_{F,Q}(U) = 2$, i.e. the unique two-dimensional $\mathbf{c}$-module. (See Lemma 5.6(b).)*

Proof. (a) A GAP calculation [GAP26] shows that the 2-permutation module $W := \mathrm{Res}_F^G(15)$ affords the character

$$\chi_{\widehat{W}} = \mathrm{Res}_F^G(\chi_{\widehat{15}}) = \chi_{17} + \chi_{16}.$$

Furthermore, [Lan83, I, Theorem 17.3] implies that W has kF-submodules U and V affording the characters χ_{16} and χ_{17}, respectively. We then read from the decomposition matrix of $\mathbf{C}$ that $V \cong 9_1$ and U is indecomposable with composition factors 3_1 and 3_3. It follows that $U \cap V = 0$ and so $W = U \oplus V$. Hence, 9_1 is a trivial source module, and P is a vertex of 9_1 as it is odd-dimensional.

Recalling that $Q \leq P \leq N_F(P) = N_G(P) \leq N_F(Q) = N_G(Q) \leq F \leq G$, we have that the number of isomorphism classes of trivial source $\mathbf{C}$-modules with vertex P equals the number of isomorphism classes of trivial source modules with vertex P lying in the Brauer correspondent $\mathbf{c}$ of $\mathbf{C}$ (which is also the Brauer correspondent of $\mathbf{A}$) at the level of $N_F(Q)$, and we have already proved in Lemma 5.6(c) that this number is one. This proves our claim.

(b) By Lemma 5.6(a) and (c), we also know that the number of isomorphism classes of trivial source $\mathbf{C}$-modules with vertex Q equals the number of isomorphism classes of trivial source $k\mathfrak{S}_4$-modules with vertex Q. This number is two, as $N_{\mathfrak{S}_4}(Q)/Q \cong \mathfrak{S}_3$, which has two conjugacy classes of $2'$-elements.

Let $f_{G,F}$ denote the Green correspondence with respect to (G,P,F) from G to F. The proof of (a) shows that $f_{G,F}(15) = 9_1$, thus, as the Green correspondence and the Brauer correspondence commute, $f_{G,F}(6)$ belongs to $\mathbf{C}$ as well. It is a trivial source module with vertex Q and, by definition, it is a direct summand of the 2-permutation module $W' := \mathrm{Res}^G_F(6)$. Moreover, $\chi_{\widehat{f_{G,F}(6)}}$ is a sum of constituents of $\chi_{\widehat{W'}} = \mathrm{Res}^G_F(\chi_{\widehat{6}})$ and by [Lan83, II, Lemma 12.6(ii)] $\chi_{\widehat{f_{G,F}(6)}}$ takes non-negative integer values at all 2-elements. Therefore, it follows from Notation 6.1(12) that the only possibility is

$$\mathrm{Res}^G_F(\chi_{\widehat{6}}) = \chi_{\widehat{f_{G,F}(6)}} = \chi_{16}\,.$$

We then read from the decomposition matrix of $\mathbf{C}$ that $f_{G,F}(6)$ is uniserial of length 2 with composition factors 3_1 and 3_3. The claim follows.

(c) Let $f_{G,Q}$ denote the Green correspondence with respect to $(G,Q,N_G(Q))$ from G to $N_G(Q)$. Since $N_F(Q) = N_G(Q)$, we have

$$f_{F,Q}(U) = f_{F,Q}(f_{G,F}(6)) = f_{G,Q}(6) = 2$$

by the proof of (b) and Lemma 5.7(c).

□

We can now prove that the functor $M \otimes_{\mathbf{C}} -$ maps the simple $\mathbf{C}$-modules to the simple $\mathbf{B}$-modules.

Lemma 6.13. *Let M be the $(\mathbf{B},\mathbf{C})$-bimodule defined in Notation 6.7(18). Then, permuting the labelling of $\tilde{3}_1$ and $\tilde{3}_2$ if necessary, the following assertions hold:*

(a) $M \otimes_{\mathbf{C}} 9_1 \cong k_H$ *as* $\mathbf{B}$*-modules and* $M^* \otimes_{\mathbf{B}} k_H \cong 9_1$ *as* $\mathbf{C}$*-modules;*

(b) $M \otimes_{\mathbf{C}} U(3_1,3_3) \cong U(\tilde{3}_1,\tilde{3}_2)$ *as* $\mathbf{B}$*-modules; and*

(c) $M \otimes_{\mathbf{C}} 3_1 \cong \tilde{3}_1$ *and* $M \otimes_{\mathbf{C}} 3_3 \cong \tilde{3}_2$ *as* $\mathbf{B}$*-modules.*

Proof. First, by [Lin96b, Theorem 2.1(ii)], the $\mathbf{B}$-modules $M \otimes_{\mathbf{C}} 9_1$, $M \otimes_{\mathbf{C}} 3_1$ and $M \otimes_{\mathbf{C}} 3_3$ are all indecomposable and non-projective, since 9_1, 3_1 and 3_3 are non-projective simple $\mathbf{C}$-modules.

(a) Since k_H is the unique trivial source $\mathbf{B}$-module with vertex P by Lemma 6.11(a) and 9_1 is the unique trivial source $\mathbf{C}$-module with vertex P by Lemma 6.12(a), it follows from Lemma 2.7 that $M \otimes_{\mathbf{C}} 9_1 \cong k_H$. An analogous argument with M^* yields the second claim.

(b) As above, write $U := U(3_1,3_3)$ and $\widetilde{U} := U(\widetilde{3}_1,\widetilde{3}_2)$. First, by Lemma 2.7(b), we have that

$$M \otimes_{\mathbf{C}} U \cong Y \oplus \mathcal{P}$$

where Y is a trivial source $\mathbf{B}$-module with vertex Q and $\mathcal{P}$ is a projective $\mathbf{B}$-module. It remains to prove that $Y \cong \widetilde{U}$ and $\mathcal{P} \cong 0$. We proceed in three steps.

(i) First, we claim that $Y = \widetilde{U}$.

Recall that $N_H(Q) \cong \mathfrak{S}_4 < N_F(Q) \cong C_3 \times \mathfrak{S}_4$ and $\mathrm{Res}^{N_F(Q)}_{N_H(Q)}$ induces a splendid Morita equivalence between the Brauer correspondent $\mathbf{b}$ of $\mathbf{B}$ in $N_H(Q)$ and the Brauer correspondent $\mathbf{c}$ of $\mathbf{C}$ in $N_F(Q)$ (see Lemma 5.6(a)). Also notice that $X = e_{\mathbf{B}}\widetilde{X}e_{\mathbf{C}} \cong e_{\mathbf{B}}kH \otimes_{kP} kFe_{\mathbf{C}} = \mathrm{Res}^{H\times H}_{H\times P}(\mathbf{B}) \otimes_{kP} \mathrm{Res}^{F\times F}_{P\times F}(\mathbf{C})$. Thus, we have

$$M \otimes_{\mathbf{C}} U \,\Big|\, \mathrm{Ind}^H_{N_H(Q)} \circ \mathrm{Res}^{N_F(Q)}_{N_H(Q)} \circ \mathrm{Res}^F_{N_F(Q)}(U)$$

and we claim that $\widetilde{U}$ is the unique direct summand with vertex Q of the right-hand side module. Considering first the Green correspondence with respect to $(F, Q, N_F(Q))$, Lemma 6.12(c) yields

$$\mathrm{Res}^F_{N_F(Q)}(U) = f_{F,Q}(U) \oplus V = 2 \oplus V$$

for some 2-permutation $kN_F(Q)$-module V. Hence, $\mathrm{Res}^F_{N_H(Q)}(U) \cong 2_{\mathfrak{S}_4} \oplus V'$ for some 2-permutation $kN_H(Q)$-module V' of dimension 4. Using again [Lan83, II, Lemma 12.6(ii)] and the character table of $\mathfrak{S}_4$, we obtain that V' is indecomposable and $\chi_{\widehat{V'}}$ takes value zero at the elements of order 4 and only one of the two conjugacy classes of elements of order 2, implying that V' has a cyclic vertex of order 2. Next, we consider the Green correspondence with respect to $(H, Q, N_H(Q))$. By Lemma 6.11(c), we have $f_{H,Q}(\widetilde{U}) = 2_{\mathfrak{S}_4}$, so $\widetilde{U}$ is the unique indecomposable summand of $\mathrm{Ind}^H_{N_H(Q)}(2_{\mathfrak{S}_4})$ with vertex Q, and $\mathrm{Ind}^H_{N_H(Q)}(V')$ does not have an indecomposable summand with vertex Q. It follows that $Y = \widetilde{U}$.

(ii) We claim that $P(k_H) \nmid \mathcal{P}$. It is enough to prove that $\mathrm{Hom}_{\mathbf{B}}(M \otimes_{\mathbf{C}} U, k_H) \cong 0$. First, as k_H is simple, by [Lin18a, Corollary 4.13.4], we have $\mathrm{Hom}_{\mathbf{B}}(M \otimes_{\mathbf{C}} U, k_H) \cong \underline{\mathrm{Hom}}_{\mathbf{B}}(M \otimes_{\mathbf{C}} U, k_H)$. Therefore, it follows from (a) that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{B}}(M \otimes_{\mathbf{C}} U, k_H) &\cong \underline{\mathrm{Hom}}_{\mathbf{B}}(M \otimes_{\mathbf{C}} U, k_H)\\ &\cong \underline{\mathrm{Hom}}_{\mathbf{C}}(U, M^* \otimes_{\mathbf{B}} k_H)\\ &\cong \underline{\mathrm{Hom}}_{\mathbf{C}}(U(3_1, 3_3), 9_1) \cong 0\,.\end{aligned}$$

(iii) We claim that $\mathcal{P} \cong 0$.
First recall from Notation 6.1(11) that $\mathrm{Irr}_k(\mathbf{B}) = \{k_H, \tilde{3}_1, \tilde{3}_2\}$. Hence, it follows from (ii) that $\mathcal{P}$ is of the form

$$\mathcal{P} = n_1 P(\tilde{3}_1) \oplus n_2 P(\tilde{3}_2)$$

where n_1, n_2 are non-negative integers. By Lemma 6.11(b) we have $\chi_{\widehat{\widetilde{U}}} = \chi_4$ and a GAP calculation [GAP26] yields

$$\mathrm{Ind}^H_{N_H(Q)} \circ \mathrm{Res}^{N_F(Q)}_{N_H(Q)} \circ \mathrm{Res}^F_{N_F(Q)}(\chi_{\widehat{U}}) = \chi_1 + 3\chi_4 + \chi_5 + 2\eta$$

where η is an irreducible character belonging to the unique block of defect zero of H. (See [WTP$^+$, L3(2)mod2].) Thus, we have

$$\chi_{\widehat{\mathcal{P}}} \mid \chi_1 + 2\chi_4 + \chi_5\,.$$

However, we read from the decomposition matrix of $\mathbf{B}$ in Notation 6.1(11) that

$$\chi_{\widehat{P(\tilde{3}_1)}} = \chi_2 + \chi_4 + \chi_5 \qquad \text{and} \qquad \chi_{\widehat{P(\tilde{3}_2)}} = \chi_3 + \chi_4 + \chi_5\,,$$

proving that $\mathcal{P} \cong 0$.

(c) The bimodule M is left and right projective, so the functor $M \otimes_{\mathbf{C}} -$ is exact. It follows that the non-split exact sequence

$$0 \longrightarrow 3_3 \xhookrightarrow{\ \iota\ } U(3_1, 3_3) \xrightarrow{\ \pi\ } 3_1 \longrightarrow 0$$

of $\mathbf{C}$-modules (where ι is the inclusion of 3_3 in the socle of $U(3_1, 3_3)$ and π the canonical projection onto its head) is mapped to the non-split exact sequence

$$0 \longrightarrow M \otimes_{\mathbf{C}} 3_3 \xrightarrow{M \otimes_{\mathbf{C}} \iota} M \otimes_{\mathbf{C}} U(3_1, 3_3) \xrightarrow{M \otimes_{\mathbf{C}} \pi} M \otimes_{\mathbf{C}} 3_1 \longrightarrow 0$$

of $\mathbf{B}$-modules, whose middle term is $M \otimes_{\mathbf{C}} U(3_1, 3_3) \cong U(\tilde{3}_1, \tilde{3}_2)$ by (b). Since the latter module has length 2, the additivity of composition lengths implies that both $M \otimes_{\mathbf{C}} 3_3$ and $M \otimes_{\mathbf{C}} 3_1$ have length 1, so they are simple and the claim follows.

□

Theorem 6.14. *The two faithful blocks* $\mathbf{C}, \mathbf{C}^* \in \mathrm{Bl}^{\max}(k[3.\mathfrak{A}_6])$ *are splendidly Morita equivalent to the principal block* $\mathbf{B} = \mathbf{B}_0(k\,\mathrm{PSL}_2(7))$ *of the projective special linear group* $\mathrm{PSL}_2(7)$.

Proof. By Proposition 6.10, the $(\mathbf{B}, \mathbf{C})$-bimodule M defined in Notation 6.7(18) realises a stable equivalence of Morita type between $\mathbf{B}$ and $\mathbf{C}$. Furthermore, by Lemma 6.13 for any simple $\mathbf{C}$-module S, the $\mathbf{B}$-module $M \otimes_{\mathbf{C}} S$ is simple. Therefore, M realises a Morita equivalence between $\mathbf{B}$ and $\mathbf{C}$ by [Lin96b, Theorem 2.1(iii)]. Since M is a 2-permutation bimodule by definition, this equivalence is splendid, as required. Replacing $\mathbf{C}$ with $\mathbf{C}^*$, a completely analogous argument proves that there is a splendid Morita equivalence between $\mathbf{C}^*$ and $\mathbf{B}$. □

7. Proof of the main results

Finally, we prove our main result.

Proof of Theorem 1.1. Let $\mathbf{B} \in \mathrm{Bl}^{\max}(kG)$. By Lemma 4.1, $\mathbf{B}$ is splendidly Morita equivalent to a block $\widehat{\mathbf{B}}$ of a central extension $\widehat{L}$ of a group L, with dihedral Sylow 2-subgroups of order 2^n and $O_{2'}(L) = 1$, by a cyclic group of order $|L|_{2'}$. Clearly, $\widehat{\mathbf{B}} \in \mathrm{Bl}^{\max}(k\widehat{L})$, and by Subsection 2.4 the group L is of type **(D1)**, **(D2)**, **(D3)** or **(D4)**.

Now, first assume that $L \notin \{\mathfrak{A}_7, \mathrm{PSL}_2(9)\}$. Then, $H^2(L, k^\times)$ is trivial by Lemma 2.2. Therefore, we can take $\widehat{L} = L$ by Theorem 2.5(c), and the claim follows from Proposition 4.2 in this case.

Next, assume that $L = \mathrm{PSL}_2(9) \cong \mathfrak{A}_6$. Let Z be the cyclic group of order $|L|_{2'} = 45$. By Lemma 2.2, we have

$$\mathrm{H}^2(L, k^\times) \cong C_3.$$

It follows that, besides the trivial extension $Z \times L$, up to isomorphism of groups, there is a unique non-split central extension H of L by Z. This extension is obtained by starting from the Schur cover

$$1 \longrightarrow C_3 \longrightarrow 3.L \longrightarrow L \longrightarrow 1$$

and pushing out along the embedding $C_3 \hookrightarrow Z$. Thus, if we write $\widetilde{L} := 3.L$, then

$$H \cong (\widetilde{L} \times Z)/\Delta(C_3),$$

that is, H is the central product of $\widetilde{L}$ and Z, obtained by identifying the central subgroup of order 3 in $\widetilde{L}$ with the unique subgroup of order 3 in Z.

If $\widehat{L} = Z \times L$, then any block $\widehat{\mathbf{B}} \in \mathrm{Bl}^{\max}(k\widehat{L})$ is splendidly Morita equivalent to $\mathbf{B}_0(kL)$ by Lemma 2.1, because the principal block of L is the unique block of maximal defect of L by Proposition 4.2. Now, if $\widehat{L} = H$, by [Düv04, Proposition 1.5] (see Subsection 2.3), every block $\widehat{\mathbf{B}} \in \mathrm{Bl}^{\max}(kH)$ is splendidly Morita equivalent to $\widetilde{\mathbf{B}} \otimes_k kZe_\lambda$, where $\widetilde{\mathbf{B}}$ is a maximal-defect block of $k\widetilde{L}$ and e_λ is a block idempotent of the split semisimple algebra kZ corresponding to an irreducible character λ of Z. Moreover, each block $\widetilde{\mathbf{B}} \otimes_k kZe_\lambda$ is splendidly Morita equivalent to $\widetilde{\mathbf{B}}$ by Lemma 2.1.

For the case $L = \mathfrak{A}_7$, we have $|L|_{2'} = 315$ and an analogous argument works replacing the cyclic group C_{45} with C_{315}.

Therefore, it remains to consider the blocks $\widetilde{\mathbf{B}} \in \mathrm{Bl}^{\max}(k\widetilde{L})$ for $\widetilde{L} \in \{3.\mathfrak{A}_6, 3.\mathfrak{A}_7\}$. First, if $\widetilde{\mathbf{B}}$ is the principal block, then there is nothing to do. Indeed, since the centre has order 3, inflation from $\widetilde{L}/Z(\widetilde{L}) \cong L$ to $\widetilde{L}$ induces a splendid Morita equivalence between $\mathbf{B}_0(kL)$ and $\mathbf{B}_0(k\widetilde{L})$. Finally, if $\widetilde{B}$ is a faithful block of $k\widetilde{L}$, then $\widetilde{\mathbf{B}}$ is splendidly Morita equivalent to $\mathbf{B}_0(k\,\mathrm{PSL}_2(7))$ by Theorem 6.14 provided $\widetilde{L} = 3.\mathfrak{A}_6$, whereas $\widetilde{\mathbf{B}}$ is splendidly Morita equivalent to $\mathbf{B}_0(k\mathfrak{S}_4)$ by Theorem refthm:FaithfulBlocks3A7 provided $\widetilde{L} = 3.\mathfrak{A}_7$. Since $\mathfrak{S}_4 \cong \mathrm{PGL}_2(3)$, the assertion is proved. This gives us all the blocks listed in cases (a)–(f).

In cases (c)–(f), any choice of q satisfying the indicated condition yields a representative of the corresponding splendid Morita equivalence class by Proposition 4.2. □

References

[AKO11] M. Aschbacher, R. Kessar, and B. Oliver, *Fusion systems in algebra and topology.*, *Lond. Math. Soc. Lect. Note Ser.* **391**, Cambridge: Cambridge University Press, 2011.

[BK06] R. Boltje and B. Külshammer, The ring of modules with endo-permutation source, *Manuscripta Math.* **120** (2006), 359–376.

[BP25] R. Boltje and P. Perepelitsky, p-permutation equivalences between blocks of group algebras, *Journal of Algebra* **664** (2025), 815–887.

[Bon11] C. Bonnafé, *Representations of* $\mathsf{SL}_2(\mathbb{F}_q)$, London: Springer, 2011.

[Bra74] R. Brauer, On 2-blocks with dihedral defect groups, Symp. math. 13, Gruppi abeliani, Gruppi e loro rappresent., Convegni 1972, 367-393, 1974.

[Bro85] M. Broué, On Scott modules and p-permutation modules: An approach through the Brauer morphism, *Proc. Am. Math. Soc.* **93** (1985).

[Bro90] M. Broué, Isométries de caractères et équivalences de Morita ou dérivées, *Inst. Hautes Études Sci. Publ. Math.* (1990), 45–63.

[CCN+85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray*, Oxford: Clarendon Press. XXXIII, 252 p. (1985)., 1985 (English).

[Cra19] D. A. Craven, *Representation theory of finite groups: a guidebook*, *Universitext*, Cham: Springer, 2019.

[CEKL11] D. A. Craven, C. W. Eaton, R. Kessar, and M. Linckelmann, The structure of blocks with a Klein four defect group, *Math. Z.* **268** (2011), 441–476.

[Düv04] O. Düvel, On Donovan's conjecture, *J. Algebra* **272** (2004), 1–26.

[Eat26] C. Eaton, Library of blocks of finite groups: Donovan's conjecture and the classification of Morita equivalence classes, (2026). Available at `https://wiki.manchester.ac.uk/blocks/index.php/Main_Page`.

[EEK+26] C. Eaton, F. Eisele, R. Kessar, M. Linckelmann, and A. A. Schaeffer Fry, On 2-blocks with quaternion defect groups, 2026. doi:10.48550/arXiv.2608.12052.

[Eis12] F. Eisele, Group rings over the p-adic integers, *Dissertation, RWTH Aachen* (2012).

[Erd90] K. Erdmann, *Blocks of tame representation type and related algebras*, **1428**, Berlin etc.: Springer-Verlag, 1990.

[Fei80] W. Feit, Irreducible modules of p-solvable groups, in *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, *Proc. Sympos. Pure Math.* **37**, Amer. Math. Soc., Providence, RI, 1980, pp. 405–411.

[Fei82] W. Feit, *The representation theory of finite groups*, *North-Holland Math. Libr.* **25**, Elsevier (North-Holland), Amsterdam, 1982.

[GAP26] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.16.1*, 2026. Available at `https://www.gap-system.org`.

[GW65] D. Gorenstein and J. H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups. I, II, III, *J. Algebra* **2** (1965), 85–151, 218–270.

[HK99] A. Hida and S. Koshitani, Morita equivalent blocks in non-normal subgroups and p-radical blocks in finite groups, *J. Lond. Math. Soc., II. Ser.* **59** (1999), 541 –556.

[HK00] G. Hiss and R. Kessar, Scopes reduction and Morita equivalence classes of blocks in finite classical groups, *J. Algebra* **230** (2000), 378–423.

[HK05] G. Hiss and R. Kessar, Scopes reduction and Morita equivalence classes of blocks in finite classical groups. II, *J. Algebra* **283** (2005), 522–563.

[HL24] G. Hiss and C. Lassueur, On the source algebra equivalence class of blocks with cyclic defect groups, I, *Beitr. Algebra Geom.* **65** (2024), 187–207.

[HL25] G. Hiss and C. Lassueur, On the source algebra equivalence class of blocks with cyclic defect groups, II, 2025. Available at `https://arxiv.org/abs/2502.09176`.

[HL26] G. Hiss and C. Lassueur, On the source algebra equivalence class of blocks with cyclic defect groups, III, 2026. Available at `https://arxiv.org/abs/2601.01582`.

[JK09] G. James and A. Kerber, *The representation theory of the symmetric group. Foreword by P. M. Cohn, introduction by G. de B. Robinson.*, reprint of the 1985 hardback ed. ed., *Encycl. Math. Appl.* **16**, Cambridge: Cambridge University Press, 2009.

[Jon74] M. R. Jones, Some inequalities for the multiplicator of a finite group, II, *Proc. Am. Math. Soc.* **45** (1974), 167—172.

[Kar87] G. Karpilovsky, *The Schur multiplier*, *Lond. Math. Soc. Monogr., New Ser.* **2**, Clarendon Press, Oxford, 1987.

[Kes00] R. Kessar, Equivalences for blocks of the Weyl groups, *Proc. Amer. Math. Soc.* **128** (2000), 337–346.

[Kes01] R. Kessar, Source algebra equivalences for blocks of finite general linear groups over a fixed field, *Manuscripta Math.* **104** (2001), 145–162.

[Kes96] R. Kessar, Blocks and source algebras for the double covers of the symmetric and alternating groups, *J. Algebra* **186** (1996), 872–933.

[Kes02] R. Kessar, Scopes reduction for blocks of finite alternating groups, *Q. J. Math.* **53** (2002), 443–454.

[KKW04] S. Koshitani, N. Kunugi, and K. Waki, Broué's Abelian defect group conjecture for the Held group and the sporadic Suzuki group., *J. Algebra* **279** (2004), 638–666.

[KL19] S. Koshitani and C. Lassueur, Splendid Morita equivalences for principal 2-blocks with dihedral defect groups, *Math. Z.* **294** (2019), 639–666.

[KL20] S. Koshitani and C. Lassueur, Splendid Morita equivalences for principal blocks with generalised quaternion defect groups, *J. Algebra* **558** (2020), 523–533.

[KLS21] S. Koshitani, C. Lassueur, and B. Sambale, Splendid Morita equivalences for principal blocks with semidihedral defect groups, *Proc. Amer. Math. Soc.* **150** (2021), 41–53.

[KMN11] S. Koshitani, J. Müller, and F. Noeske, Broué's Abelian defect group conjecture holds for the sporadic simple Conway group Co_3., *J. Algebra* **348** (2011), 354–380. doi:10.1016/j.jalgebra.2011.10.001.

[KS23] N. Kunugi and K. Suzuki, Relative stable equivalences of Morita type for the principal blocks of finite groups and relative Brauer indecomposability, *J. Group Theory* **26** (2023), 1157–1184.

[Lan83] P. Landrock, *Finite group algebras and their modules*, *Lond. Math. Soc. Lect. Note Ser.* **84**, Cambridge University Press, Cambridge. London Mathematical Society, London, 1983.

[Las23] C. Lassueur, A tour of p-permutation modules and related classes of modules, *Jahresber. Dtsch. Math.-Ver.* **125** (2023), 137–189.

[Lin96a] M. Linckelmann, The isomorphism problem for cyclic blocks and their source algebras, *Invent. Math.* **125** (1996), 265–283.

[Lin96b] M. Linckelmann, Stable equivalences of Morita type for self-injective algebras and p-groups, *Math. Z.* **223** (1996), 87–100.

[Lin01] M. Linckelmann, On splendid derived and stable equivalences between blocks and finite groups, *J. Algebra* **242** (2001), 819–843.

[Lin18a] M. Linckelmann, *The block theory of finite group algebras. Vol. I*, *London Mathematical Society Student Texts* **91**, Cambridge University Press, Cambridge, 2018.

[Lin18b] M. Linckelmann, *The block theory of finite group algebras. Vol. II*, *London Mathematical Society Student Texts* **92**, Cambridge University Press, Cambridge, 2018.

[LMF26] LMFDB Collaboration, The L-functions and modular forms database, `https://www.lmfdb.org`, 2026.

[Mac22] N. Macgregor, Morita equivalence classes of tame blocks of finite groups, *J. Algebra* **608** (2022), 719–754.

[NT89] H. Nagao and Y. Tsushima, Representations of finite groups. Transl. from the Japanese, Boston, MA etc.: Academic Press, 1989.

[NT12] G. Navarro and P. H. Tiep, Brauer's height zero conjecture for the 2-blocks of maximal defect, *J. Reine Angew. Math.* **669** (2012), 225–247.

[Oku81] T. Okuyama, Module correspondence in finite groups, *Hokkaido Math. J.* **10** (1981), 299–318.

[Ope26a] OpenAI, Donovan's conjecture over algebraically closed fields, Preprint, September 2026, Preprint, 24 September 2026. Available at `https://github.com/openai/math/tree/main/preprints/Donovans-Conjecture-over-Algebraic-Closures-of-Prime-Fields-September-24-2026`.

[Ope26b] OpenAI, Integral Donovan finiteness over Witt vectors, September 2026, Preprint, 25 September 2026. Available at `https://github.com/openai/math/tree/main/preprints/Integral-Donovan-Finiteness-over-Witt-Vectors-September-25-2026`.

[Pui94] L. Puig, On Joanna Scopes' criterion of equivalence for blocks of symmetric groups, *Algebra Colloq.* **1** (1994), 25–55.

[Pui88] L. Puig, Nilpotent blocks and their source algebras, *Invent. Math.* **93** (1988), 77–116.

[Sal08] A. Salminen, On the sources of simple modules in nilpotent blocks, *J. Algebra* **319** (2008), 4559–4574.

[Sam14] B. Sambale, *Blocks of finite groups and their invariants*, *Lecture Notes in Mathematics* **2127**, Springer, Cham, 2014.

[Thé95] J. Thévenaz, *G-algebras and modular representation theory*, *Oxford Mathematical Monographs*, The Clarendon Press, Oxford University Press, New York, 1995, Oxford Science Publications.

[Thé07] J. Thévenaz, Endo-permutation modules, a guided tour, in *Group representation theory*, EPFL Press, Lausanne, 2007, pp. 115–147.

[Wei94] C. A. Weibel, *An introduction to homological algebra*, *Camb. Stud. Adv. Math.* **38**, Cambridge: Cambridge University Press, 1994.

[WTP$^+$] R. Wilson, J. Thackray, R. Parker, F. Noeske, J. Müller, F. Lübeck, C. Jansen, G. Hiss, and T. Breuer, The modular atlas project. Available at `http://www.math.rwth-aachen.de/~MOC`.

A. Proof of the vertex and source statement in Fong's second reduction

In this Appendix, we give a proof of Part (b) of Theorem 2.5 establishing that Fong's second reduction can be understood as a Morita equivalence with endo-permutation source. This result, understood in terms of source-algebras, is due to Puig and we refer to [Lin18b, Theorem 9.11.9], which states the connection between the source-algebras of the blocks involved and the source of the bimodule giving the Morita equivalence. Below we give a precise proof exhibiting a source and a vertex of this bimodule purely in module-theoretic terms, which in particular does not require knowledge of the source-algebras.

Proof of Theorem 2.5, Part (b) *and Remark 2.6.*

(i) *The module V and its restriction to D.* Since $N \cap D = 1$, the quotient group $ND/N \cong D$ is a p-group. Moreover, since $\mathbf{c}$ is G-stable, the inertia group of the simple $\mathbf{c}$-module V in ND is ND. Therefore, by [Fei82, Chapter III, Corollary 3.16], V has a unique extension, up to isomorphism, to an irreducible $k[ND]$-module. We denote this extension again by V and set $W := \mathrm{Res}_D^{ND}(V)$. Consider $S = kNe_{\mathbf{c}}$ as a kD-module via conjugation. Then

$$S = kNe_{\mathbf{c}} \cong \mathrm{End}_k(V) \cong \mathrm{End}_k(W)$$

as kD-modules. Since S is a direct summand of the permutation kD-module kN, it follows that $\mathrm{End}_k(W)$ is a permutation kD-module. Hence W is an endo-permutation kD-module.

Moreover, $\mathrm{Br}_D(e_{\mathbf{c}}) \neq 0$, and hence $\mathrm{End}_k(V) \cong kNe_{\mathbf{c}}$ has a trivial direct summand as a kD-module. It follows from [Thé07, Lemma 2.2] that W is capped. We set

$$U := \mathrm{cap}(W).$$

Thus U is an indecomposable kD-module with vertex D.

We next show that the kND-module V has vertex D. Let Q be a vertex of V. Since $\mathbf{c}$ is a block of defect zero, the module V belongs to a block of kND having a defect group P such that $P \cap N = 1$. Therefore, $|Q| \leq |P| \leq |D|$. On the other hand, U is a direct summand of

$\mathrm{Res}_D^{ND}(V)$ and has vertex D. Hence D is contained, up to conjugation, in a vertex of V. It follows that $|D| \leq |Q|$. Therefore $|Q| = |D|$, and D is a vertex of V.

(ii) *Construction of the Morita equivalence.* By [Sal08, Section 3], there exists an isomorphism

$$\Phi : S \otimes_k kHe \xrightarrow{\sim} kGe_{\mathbf{c}}$$

of interior D-algebras, where e is an idempotent of kZ. Here, we identify the copy of D in H arising from this construction with D.

Since S is simple, there is a bijection between the block idempotents of kH appearing in e and the blocks of kG appearing in $e_{\mathbf{c}}$; see [Sal08, Section 4]. Let $\widehat{\mathbf{B}}$ be the block of kH corresponding to $\mathbf{B}$. By the proof of [Sal08, Proposition 4.1], which does not use the nilpotency assumption, the chosen copy of D in H is a defect group of $\widehat{\mathbf{B}}$.

The isomorphism Φ restricts to an isomorphism $\Phi_{\mathbf{B}} : S \otimes_k \widehat{\mathbf{B}} \xrightarrow{\sim} \mathbf{B}$ of interior D-algebras. Since $S \cong \mathrm{End}_k(V)$, the $(S \otimes_k \widehat{\mathbf{B}}, \widehat{\mathbf{B}})$-bimodule $V \otimes_k \widehat{\mathbf{B}}$ induces a Morita equivalence between $S \otimes_k \widehat{\mathbf{B}}$ and $\widehat{\mathbf{B}}$. Its actions are given by

$$(s \otimes a) \cdot (v \otimes x) \cdot a' = s(v) \otimes axa'$$

for all $s \in S$, $a, a', x \in \widehat{\mathbf{B}}$ and $v \in V$; see [Thé95, Exercise 9.5(a)]. Transporting the left action along $\Phi_{\mathbf{B}}^{-1}$, the bimodule $M := V \otimes_k \widehat{\mathbf{B}}$ becomes a $(\mathbf{B}, \widehat{\mathbf{B}})$-bimodule and induces a Morita equivalence between $\mathbf{B}$ and $\widehat{\mathbf{B}}$.

(iii) *Relative projectivity of M.* Set $\overline{G} := G/N$ and let $\pi : H \to \overline{G}$ be the canonical surjection. Consider the $(kGe_{\mathbf{c}}, kHe)$-bimodule $\mathcal{M} := V \otimes_k kHe$, where the left action is transported along $\Phi : S \otimes_k kHe \xrightarrow{\sim} kGe_{\mathbf{c}}$. Recall that Z is identified with a finite subgroup of $k^\times$ and that

$$e = \frac{1}{|Z|} \sum_{\zeta \in Z} \zeta^{-1}(1, \zeta).$$

For every $\zeta_0 \in Z$, we have $(1, \zeta_0)e = \zeta_0 e$, where, on the right-hand side, ζ_0 is regarded as an element of $k^\times$. For every $\overline{x} \in \overline{G}$, choose an element $\widetilde{x} \in H$ with $\pi(\widetilde{x}) = \overline{x}$. Since Z is central in H, we have

$$(1, \zeta)\widetilde{x}e = \widetilde{x}(1, \zeta)e = \zeta \widetilde{x}e \qquad (\zeta \in Z).$$

It follows that $kZ\widetilde{x}e = k\widetilde{x}e$. Therefore, as a k-vector space,

$$kHe = \bigoplus_{\overline{x} \in \overline{G}} k\widetilde{x}e,$$

and hence

$$\mathcal{M} = \bigoplus_{\overline{x} \in \overline{G}} \mathcal{M}_{\overline{x}} \quad \text{with} \quad \mathcal{M}_{\overline{x}} := V \otimes_k k\widetilde{x}e.$$

We now use the explicit construction of the same isomorphism Φ. By [Sal08, Section 3], for every $g \in G$ there exist $s_g \in S^\times$ and $\alpha_g \in k^\times$ such that

$$\Phi^{-1}(ge_{\mathbf{c}}) = s_g \otimes \alpha_g \widetilde{gN} e, \tag{1}$$

where $\widetilde{gN} \in H$ is a lift of $gN \in \overline{G}$. Let $\overline{x} \in \overline{G}$, $v \otimes y \in \mathcal{M}_{\overline{x}}$ and $y \in k\widetilde{x}e$. Using (1), we obtain

$$(g, h) \cdot (v \otimes y) = g \cdot (v \otimes y) \cdot h^{-1} = \alpha_g s_g(v) \otimes \widetilde{gN} y h^{-1}.$$

The image of $\widetilde{gN}\widetilde{x}h^{-1}$ under π is $(gN)\overline{x}\pi(h)^{-1}$. Therefore $(g, h)\mathcal{M}_{\overline{x}} \subseteq \mathcal{M}_{(gN)\overline{x}\pi(h)^{-1}}$. Since the map induced by (g, h) is invertible and both spaces have dimension $\dim_k(V)$, we obtain

$$(g, h)\mathcal{M}_{\overline{x}} = \mathcal{M}_{(gN)\overline{x}\pi(h)^{-1}}. \tag{2}$$

In particular, $(g,h)\mathcal{M}_{\overline{1}} = \mathcal{M}_{\overline{1}}$ if and only if $gN = \pi(h)$. Thus the stabilizer of $\mathcal{M}_{\overline{1}} = V \otimes_k ke$ in $G \times H$ is

$$L := \{(g,h) \in G \times H \mid gN = \pi(h)\}.$$

Moreover, Equation (2) shows that $G \times H$ acts transitively on $\left\{\mathcal{M}_{\overline{x}} \;\middle|\; \overline{x} \in \overline{G}\right\}$. It follows that the natural homomorphism

$$\Theta : \mathrm{Ind}_L^{G\times H}(\mathcal{M}_{\overline{1}}) \longrightarrow \mathcal{M}, \qquad (g,h) \otimes m \longmapsto (g,h) \cdot m,$$

is surjective. Since both sides have the same dimension, Θ is an isomorphism, so $\mathcal{M}$ is induced from $\mathcal{M}_{\overline{1}}$. Since $\widehat{\mathbf{B}}$ is a block summand of kHe, the bimodule M is a direct summand of $\mathcal{M}$ and is therefore relatively L-projective.

(iv) *Vertex and source of M.* Since M induces a Morita equivalence between the block algebras $\mathbf{B}$ and $\widehat{\mathbf{B}}$, it is indecomposable as a $k[G \times H]$-module. Let R be a vertex of M. Since M is relatively L-projective, R is contained in a conjugate of L. On the other hand, M belongs to the block $\mathbf{B} \otimes_k \widehat{\mathbf{B}}^*$ of $k[G \times H]$. Hence, after replacing R by a conjugate, we may assume that $R \leq (D \times D) \cap {}^yL$ for some $y \in G \times H$.

Since $D \cap N = 1$ and $D \cap Z = 1$, it follows from the definition of L that $(D \times D) \cap {}^yL$ is a twisted diagonal subgroup of $D \times D$. In particular, $|R| \leq |D|$.

Viewed as a $k[H \times H]$-module, the regular bimodule $\widehat{\mathbf{B}}$ has vertex ΔD and trivial source. Therefore $k \mid \mathrm{Res}^{H\times H}_{\Delta D}(\widehat{\mathbf{B}})$. Since $\Phi_{\mathbf{B}}$ is an isomorphism of interior D-algebras, the action of ΔD on M is given by

$$(d,d) \cdot (v \otimes x) = dv \otimes dxd^{-1}.$$

Therefore,

$$\mathrm{Res}^{G\times H}_{\Delta D}(M) \cong W \otimes_k \mathrm{Res}^{H\times H}_{\Delta D}(\widehat{\mathbf{B}})$$

as $k[\Delta D]$-modules, where we identify $D \cong \Delta D$. It follows that $W \mid \mathrm{Res}^{G\times H}_{\Delta D}(M)$. Since $U \mid W$, we obtain $U \mid \mathrm{Res}^{G\times H}_{\Delta D}(M)$. Viewed as a $k[\Delta D]$-module, U has vertex ΔD. Therefore, there exists $z \in G \times H$ such that $\Delta D \leq {}^zR$. Hence $|D| \leq |R|$. Together with the preceding inequality, this implies that $|R| = |D|$ and ${}^zR = \Delta D$. Thus, ΔD is a vertex of M.

Finally, the indecomposable $k[\Delta D]$-module U is a direct summand of $\mathrm{Res}^{G\times H}_{\Delta D}(M)$ and has vertex ΔD. Therefore, U is a source of M. In other words, via the identification $D \cong \Delta D$, the cap of $\mathrm{Res}^{ND}_{D}(V)$ is a source of the bimodule inducing the Morita equivalence. □

Shigeo Koshitani, Department of Mathematics and Informatics, Graduate School of Science, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan
Email address: koshitan@math.s.chiba-u.ac.jp

Caroline Lassueur, Leibniz Universität Hannover, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Welfengarten 1, 30167 Hannover, Germany
Email address: lassueur@mathematik.uni-kl.de

İpek Tuvay, Mimar Sinan Fine Arts University, Department of Mathematics, 34380 Bomonti, İstanbul, Turkey
Email address: ipek.tuvay@msgsu.edu.tr

Deniz Yilmaz, Department of Mathematics, Bilkent University Ankara, Turkey
Email address: d.yilmaz@bilkent.edu.tr