On Alperin's conjecture and functorial equivalence of blocks

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Abstract

Let k be an algebraically closed field of positive characteristic p and let $\mathbb{F}$ be an algebraically closed field of characteristic 0. We consider Alperin's weight conjecture (over k) from the point of view of (stable) functorial equivalence of blocks over $\mathbb{F}$. We formulate a functorial version of Alperin's blockwise weight conjecture, and show that it is equivalent to the original one. We also show that this conjecture holds *stably*, i.e. in the category of stable diagonal p-permutation functors over $\mathbb{F}$.

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1 Introduction

Let k be an algebraically closed field of positive characteristic p. The first step in studying the representation theory of a finite group G over k consists in splitting the group algebra kG as a direct product of the *block algebras* kGb, where b is a *block idempotent* of kG, that is, a primitive idempotent of the center of kG. In this paper, such a pair (G, b) of a finite group G and a block idempotent b of kG is called a *group-block* pair (over k).

To speak very loosely, block theory is the study of such group-block pairs (G, b), and of the numerous invariants and structures attached to them: The block algebra kGb itself, in the first place, and its various - derived or stable - categories of modules. But also invariants of a more combinatorial nature, such as defect groups, source algebras or fusion systems. For each of these invariants, there is a corresponding notion of equivalence of group-block pairs: Morita equivalence, splendid Rickard equivalence, Puig equivalence, stable equivalence of Morita type, p-permutation equivalence, isotypy, perfect isometry ([9], [8], [2], [6])... All these equivalences have been the object of intense research, and a huge amount of literature is now devoted to them.

These invariants also led to important conjectures, of various kinds. Some of them, called *finiteness conjectures*, predict that there are only *finitely many* group-block pairs up to some particular equivalence, once a specific invariant is given: For example, Donovan's conjecture ([8, Conjecture 6.1.9]) states that there are only finitely many block algebras of group-block pairs with a given defect group, up to Morita equivalence. Other types of conjectures - called *local-global* or *counting conjectures* - assert that some invariants for a group-block pair (G, b) can be computed from the invariants attached to "smaller" pairs and "*local data*" - generally obtained by applying the Brauer morphism. This is in particular the case of Alperin's weight conjecture, in its blockwise form ([1]). Finally, some *structural conjectures*, like Broué's abelian defect group conjecture, predict the existence of some specific equivalence between derived categories of modules associated to group-block pairs related by the Brauer correspondence ([6]). All these conjectures have been verified in numerous cases, and proved for some particular classes of group-block pairs, but they remain essentially open.

Further new invariants were attached recently ([4], [5]) to group-block pairs, in the form of *(stable) diagonal p-permutation functors* over some commutative ring R (see Section 2 for details), leading to a corresponding notion of *(stable) functorial equivalence* of group-block pairs over R. These invariants lead in particular to a finiteness theorem ([4], Theorem 10.6), in the spirit of Donovan's conjecture: If $\mathbb{F}$ is an algebraically closed field of characteristic 0, and D is a finite p-group, then there is only a finite number of group-block pairs (G, b) with defect groups isomorphic to D, up to functorial equivalence over $\mathbb{F}$.

In the present paper, we consider Alperin's weight conjecture from the point of view of (stable) diagonal p-permutation functors. In Section 3, we first formulate (Conjecture 3.3) a functorial form $\mathrm{FAwc}(G, b)$ of Alperin's conjecture $\mathrm{Awc}(G, b)$, for each group-block pair (G, b). The main theorem of this section is Theorem 3.5, stating that Conjecture $\mathrm{FAwc}(G, b)$ holds *stably*, i.e. in the Grothendieck group of the category of stable diagonal p-permutation functors over $\mathbb{F}$. As a consequence, our conjecture $\mathrm{FAwc}(G, b)$ is in fact *equivalent* to $\mathrm{Awc}(G, b)$.

It has been suspected by many experts that the original formulation of Alperin's weight conjecture, namely the equality of two numbers associated to (G, b), is only the shadow of a more structural (yet hidden) phenomenon. We hope that $\mathrm{FAwc}(G, b)$ is a first step towards a more structural explanation.

1.1 Notation Let G be a group. By $i_g\colon G \to G$, we denote the conjugation map $x \mapsto gxg^{-1}$. Moreover, for $x \in G$ and $H \leqslant G$ we set ${}^g x := i_g(x)$ and ${}^g H := i_g(H)$. If X is a left G-set, the stabilizer of an element $x \in X$ is denoted by G_x. By $G\backslash X$ we denote the set of G-orbits of X, and by $[G\backslash X]$ we denote a set of representatives of the G-orbits of X.

2 Review of (stable) functorial equivalence of blocks

2.1. Let R be a commutative ring (with 1), and k be an algebraically closed field of characteristic $p > 0$. We denote by $\mathsf{R}pp_k^\Delta$ the category introduced in [3], where objects are finite groups, and the set of morphisms from a group G to a group H is $\mathsf{R}T^\Delta(H, G) =$

$\mathsf{R} \otimes_{\mathbb{Z}} T^\Delta(H,G)$, where $T^\Delta(H,G)$ is the Grothendieck group of *diagonal p-permutation (kH,kG)-bimodules.*

A *diagonal p-permutation functor* is by definition (see [3]) an R-linear functor from $\mathsf{R}pp_k^\Delta$ to the category R-Mod of all R-modules. These functors, together with their natural transformations, form an abelian category which we simply denote by $\mathcal{F}_{\mathsf{R},k}$ (instead of $\mathcal{F}^\Delta_{\mathsf{R}pp_k}$ as in [3]).

2.2. We denote by $\overline{\mathsf{R}pp_k^\Delta}$ the quotient category of $\mathsf{R}pp_k^\Delta$ by the morphisms that factor through the trivial group. A *stable diagonal p-permutation functor* (see [5]) is an R-linear functor from $\overline{\mathsf{R}pp_k^\Delta}$ to R-Mod, or equivalently, a diagonal p-permutation functor which vanishes at the trivial group. Stable diagonal p-permutation functors also form an abelian category, that we simply denote by $\overline{\mathcal{F}_{\mathsf{R},k}}$. If F is a diagonal p-permutation functor, we denote by $\overline{F}$ its largest stable quotient, i.e. the quotient of F by the subfunctor generated by $F(\mathbf{1})$. In other words $\overline{F}(G) = F(G)/\mathsf{R}T^\Delta(G,\mathbf{1})F(\mathbf{1})$ for any finite group G ([5], Remark 3.4).

2.3. Let (G,b) be a pair of a finite group G and a central idempotent b of kG (recall that when b is moreover *primitive*, the pair (G,b) is called a *group-block* pair). Then the (isomorphism class of the) (kG,kG)-bimodule kGb is an idempotent endomorphism of G in $\mathsf{R}pp_k^\Delta$. The diagonal p-permutation functor $\mathsf{R}T^\Delta_{G,b}$ *associated to* (G,b) is the corresponding direct summand $\mathsf{R}T^\Delta(-,G)\circ kGb$ of the representable functor at G obtained by precomposition with kGb.

We say that two group-block pairs (G,b) and (H,c) are *functorially equivalent over* R ([4], Definition 10.1) if the functors $\mathsf{R}T^\Delta_{G,b}$ and $\mathsf{R}T^\Delta_{H,c}$ are isomorphic in $\mathcal{F}_{\mathsf{R},k}$. Similarly, we say that (G,b) and (H,c) are *stably functorially equivalent over* R if the functors $\overline{\mathsf{R}T^\Delta_{G,b}}$ and $\overline{\mathsf{R}T^\Delta_{H,c}}$ are isomorphic in $\overline{\mathcal{F}_{\mathsf{R},k}}$.

2.4. We denote by $\mathsf{R}\mathcal{B}\ell_k$ the (partial[1]) idempotent completion of $\mathsf{R}pp_k^\Delta$ constructed from blocks of group algebras, i.e. the category defined as follows:

- The objects of $\mathsf{R}\mathcal{B}\ell_k$ are pairs (G,b), where G is a finite group and b is a central idempotent of kG.
- The set of morphisms from (G,b) to (H,c) in $\mathsf{R}\mathcal{B}\ell_k$ is the subset of $\mathsf{R}T^\Delta(H,G)$ obtained by precomposition with kGb and postcomposition with kHc, in other words the set $kHc \circ \mathsf{R}T^\Delta(H,G) \circ kGb$.
- The composition of morphisms $u:(G,b)\to(H,c)$ and $v:(H,c)\to(K,d)$ in $\mathsf{R}\mathcal{B}\ell_k$ is induced by the tensor product of bimodules over kH.

[1]In the classical definition of the idempotent completion of a category $\mathcal{C}$, the objects are *all* the pairs (X,e) of an object X of $\mathcal{C}$ and an idempotent endomorphism e of X in $\mathcal{C}$. Here we consider only *some* of these pairs. This does not affect the main properties of the idempotent completion.

- The identity morphism of (G,b) in $\mathsf{R}\mathcal{B}\ell_k$ is (the isomorphism class of) the (kG,kG)-bimodule kGb.

Note that the objects $(G,0)$ in $\mathsf{R}\mathcal{B}\ell_k$ are 0-objects. Similarly, we denote by $\overline{\mathsf{R}\mathcal{B}\ell_k}$ the quotient category of $\mathsf{R}\mathcal{B}\ell_k$ by the morphisms which factor through the trivial pair $(\mathbf{1},1_k)$. Equivalently $\overline{\mathsf{R}\mathcal{B}\ell_k}$ is the partial idempotent completion built as above from the category $\overline{\mathsf{R}pp_k^\Delta}$.

We denote by $\mathcal{FB}\ell_{\mathsf{R},k}$ (resp. $\overline{\mathcal{FB}\ell_{\mathsf{R},k}}$) the category of R-linear functors from $\mathsf{R}\mathcal{B}\ell_k$ (resp. $\overline{\mathsf{R}\mathcal{B}\ell_k}$) to R-Mod. By standard results, the inclusion functor $\mathsf{R}pp_k^\Delta \to \mathsf{R}\mathcal{B}\ell_k$ (resp. $\overline{\mathsf{R}pp_k^\Delta} \to \overline{\mathsf{R}\mathcal{B}\ell_k}$) sending a group G to the pair $(G,1_{kG})$ induces by composition an equivalence of categories from $\mathcal{FB}\ell_{\mathsf{R},k}$ to $\mathcal{F}_{\mathsf{R},k}$ (resp. from $\overline{\mathcal{FB}\ell_{\mathsf{R},k}}$ to $\overline{\mathcal{F}_{\mathsf{R},k}}$). We observe that if a central idempotent b of kG is an orthogonal sum of two central idempotents b_1 and b_2, the functor $\mathsf{R}T_{G,b}^\Delta$ is naturally isomorphic to the direct sum $\mathsf{R}T_{G,b_1}^\Delta \oplus \mathsf{R}T_{G,b_2}^\Delta$ in $\mathcal{F}_{R,k}$. We also observe that two group-block pairs (G,b) and (H,c) are functorially equivalent (resp. stably functorially equivalent) over R if and only if (G,b) and (H,c) are isomorphic in $\mathsf{R}\mathcal{B}\ell_k$ (resp. in $\overline{\mathsf{R}\mathcal{B}\ell_k}$).

2.5 Let $\mathbb{F}$ be an algebraically closed field of characteristic 0. It was shown in [4] and [5] that the categories $\mathcal{F}_{\mathbb{F},k}$ and $\overline{\mathcal{F}_{\mathbb{F},k}}$ are semisimple, and it follows that the categories $\mathcal{FB}\ell_{\mathbb{F},k}$ and $\overline{\mathcal{FB}\ell_{\mathbb{F},k}}$ are also semisimple. We denote by $K_0(\mathcal{FB}\ell_{\mathbb{F},k})$ and $K_0(\overline{\mathcal{FB}\ell_{\mathbb{F},k}})$ their respective Grothendieck groups.

The simple diagonal p-permutation functors over $\mathbb{F}$ are parametrized ([4], Corollary 6.15) by means of D^Δ-*pairs*: By definition ([4], Definition 3.2), this is a pair (L,u) of a finite p-group L and a generator u of a p'-group acting faithfully on L - in other words u is not only a p'-element acting on L, but a p'-*automorphism* of L. An isomorphism $\varphi : (L,u) \to (M,v)$ of D^Δ-pairs is a group isomorphism $\varphi : L\rtimes\langle u\rangle \to M\rtimes\langle v\rangle$ between the corresponding semidirect products, which sends u to a conjugate of v. We denote by $\mathrm{Aut}(L,u)$ the group of automorphisms of the pair (L,u), and by $\mathrm{Out}(L,u)$ the quotient of $\mathrm{Aut}(L,u)$ by the subgroup $\mathrm{Inn}(L\rtimes\langle u\rangle)$ of inner automorphisms of $L\rtimes\langle u\rangle$.

The simple diagonal p-permutation functors $S_{L,u,V}$ over $\mathbb{F}$ (up to isomorphism) are then parametrized by triples (L,u,V) (up to isomorphism), where (L,u) is a D^Δ-pair, and V is a simple $\mathbb{F}\mathrm{Out}(L,u)$-module. The simple stable diagonal p-permutation functors are the functors $S_{L,u,V}$, where L is a *nontrivial* p-group. So the group $K_0(\mathcal{FB}\ell_{\mathbb{F},k})$ has a $\mathbb{Z}$-basis consisting of the isomorphism classes $[S_{L,u,V}]$ of simple functors $S_{L,u,V}$, and $K_0(\overline{\mathcal{FB}\ell_{\mathbb{F},k}})$ has a $\mathbb{Z}$-basis consisting of the classes $[S_{L,u,V}]$ with $L \neq \mathbf{1}$. Any diagonal p-permutation functor F has an image in $K_0(\mathcal{FB}\ell_{\mathbb{F},k})$ (resp. in $K_0(\overline{\mathcal{FB}\ell_{\mathbb{F},k}})$), and two diagonal p-permutation functors are isomorphic (resp. stably isomorphic) if and only if they have the same image, i.e. if the multiplicity of each simple functor $S_{L,u,V}$ (resp. $S_{L,u,V}$ with $L \neq \mathbf{1}$) in both of them is the same. In particular, two group-block pairs (G,b) and (H,c) are functorially equivalent (resp. stably functorially equivalent) if and only if the multiplicities of each $S_{L,u,V}$ (resp. each $S_{L,u,V}$ with $L \neq \mathbf{1}$) in $\mathbb{F}T_{G,b}^\Delta$ and $\mathbb{F}T_{H,c}^\Delta$ are equal.

When G is a finite group, and b is a central idempotent of kG, we simply denote by $[\![G,b]\!]_{\mathbb{F}}$ the image of the functor $\mathbb{F}T^{\Delta}_{G,b}$ in $K_0(\mathcal{FB}\ell_{\mathbb{F},k})$, and by $\overline{[\![G,b]\!]_{\mathbb{F}}}$ its image in $K_0(\overline{\mathcal{FB}\ell_{\mathbb{F},k}})$.

3 Alperin's weight conjecture and stable functorial equivalence

By Knörr-Robinson, see [7, Theorem 3.8], Alperin's blockwise weight conjecture, that we refer to as AWC, is equivalent to saying that the following conjecture $\mathrm{Awc}(G,b)$ holds for all group-block pairs (G,b).

3.1 Conjecture Let (G,b) be a group-block pair over k. Then

$$(\mathrm{Awc}(G,b)) \qquad \sum_{\sigma\in[G\backslash\mathcal{S}_p(G)]}(-1)^{|\sigma|}\,l(kG_\sigma b_\sigma)=\begin{cases}1 & \text{if } d(b)=0;\\ 0 & \text{if } d(b)>0.\end{cases}$$

Here:

- $\mathcal{S}_p(G)$ denotes the set of strictly ascending chains $(\mathbf{1}=P_0<P_1<\cdots<P_n)$ of p-subgroups of G.
- G_σ is the stabilizer in G of σ.
- $|\sigma|=n$ if $\sigma=(\mathbf{1}=P_0<P_1<\cdots<P_n)\in\mathcal{S}_p(G)$.
- For σ as above, $b_\sigma:=\mathrm{Br}_{P_n}(b)$, where Br_{P_n} is the Brauer homomorphism with respect to P_n.
- l associates to a finite-dimensional k-algebra the number of its simple modules (up to isomorphism).
- $d(b)$ denotes the defect of b.

3.2 Remark One can show that b_σ is actually a sum of block idempotents of kG_σ and that it does not depend on the chain σ but only on the stabilizer G_σ (see Lemma 3.1 and the following Remark in [7]). If b is a block of kG with trivial defect group, Conjecture $\mathrm{Awc}(G,b)$ holds trivially.

For a group-block pair (G,b), we propose the following conjecture, denoted by $\mathrm{FAwc}(G,b)$ (with "F" standing for functorial), using the notation of Conjecture 3.1:

3.3 Conjecture Let (G, b) be a group-block pair over k. Then, in $K_0(\mathcal{FB}\ell_{\mathbb{F},k})$, we have

$$(\mathrm{FAwc}(G,b)) \qquad \sum_{\sigma\in[G\backslash\mathcal{S}_p(G)]} (-1)^{|\sigma|}\, [\![G_\sigma, b_\sigma]\!]_{\mathbb{F}} = \begin{cases} [S_{\mathbf{1},1,\mathbb{F}}] & \text{if } d(b)=0; \\ 0 & \text{if } d(b)>0. \end{cases}$$

Again, if b has trivial defect group, it is easy to show that Conjecture FAwc(G, b) holds, see [4, Corollary 8.23]. The statement that FAwc(G, b) holds for all group-block pairs (G, b) is abbreviated by FAWC.

It is straightforward to see that for a group-block pair (G, b), Conjecture FAwc(G, b) implies Conjecture Awc(G, b):

3.4 Theorem *1. Let (G, b) be a group-block pair over k. If Conjecture* FAwc(G, b) *holds, then Conjecture* Awc(G, b) *holds.*

2. In particular, Conjecture FAWC *implies Conjecture* AWC.

Proof Since $\mathbb{F}T^{\Delta}_{G,b}(\mathbf{1})$ is isomorphic to the $\mathbb{F}$-vector space spanned by the indecomposable projective kGb-modules, one has $l(kGb) = \dim_{\mathbb{F}} \mathbb{F}T^{\Delta}_{G,b}(\mathbf{1})$, and by [4, Corollary 8.23] the latter is equal to the multiplicity of the simple functor $S_{\mathbf{1},1,\mathbb{F}}$ in $\mathbb{F}T^{\Delta}_{G,b}$. Assertion 1 is now immediate by considering the multiplicity of $[S_{\mathbf{1},1,\mathbb{F}}]$ in both sides of FAwc(G, b), and then Assertion 2 follows. □

In the following theorem, the notation is the same as in Conjecture 3.1:

3.5 Theorem *1. Let (G, b) be a group-block pair over k. Then there exists an integer $n_{G,b}$ such that, in $K_0(\mathcal{FB}\ell_{\mathbb{F},k})$,*

$$\sum_{\sigma\in[G\backslash\mathcal{S}_p(G)]} (-1)^{|\sigma|}[\![G_\sigma, b_\sigma]\!]_{\mathbb{F}} = n_{G,b}[S_{\mathbf{1},1,\mathbb{F}}].$$

2. In particular, Conjecture FAWC *is equivalent to* AWC*, and for any group-block pair (G, b) over k, Conjecture* FAwc*(G,b) is equivalent to* Awc*(G,b).*

Proof 1. Let $\Sigma_{G,b} \in K_0(\mathcal{FB}\ell_{\mathbb{F},k})$ denote the alternating sum in the left hand side of FAwc(G, b), that is,

$$\Sigma_{G,b} := \sum_{\sigma\in[G\backslash\mathcal{S}_p(G)]} (-1)^{|\sigma|}[\![G_\sigma, b_\sigma]\!]_{\mathbb{F}}\,.$$

We want to show that the multiplicity of a simple functor $S_{L,u,V}$ in $\Sigma_{G,b}$ is equal to 0 if $L \neq 1$. By [4, Theorem 8.22], we know that the multiplicity $m_{L,u,V}(G_\sigma, b_\sigma)$ of $S_{L,u,V}$ as a composition factor of the functor $\mathbb{F}T^{\Delta}_{G,b}$ is given by

$$m_{L,u,V}(G_\sigma, b_\sigma) = \sum_{(P_\gamma,\pi)\in[G_\sigma\backslash\mathcal{L}_{b_\sigma}(G_\sigma,L,u)/\mathrm{Aut}(L,u)]} \dim_{\mathbb{F}} V^{\mathrm{Aut}(L,u)_{\overline{(P_\gamma,\pi)}}},$$

where the notation is as follows:

- $\mathcal{L}_{b_\sigma}(G_\sigma, L, u)$ is the set of pairs (P_γ, π) of a local point P_γ on $kG_\sigma b_\sigma$, i.e., a p-subgroup P of G_σ and a conjugacy class γ of primitive idempotents of $(kG_\sigma b_\sigma)^P$, and $\pi : L \to P$ is a group isomorphism such that $\pi u \pi^{-1} \in N_{G_\sigma}(P_\gamma)$, i.e., such that there exists $g \in N_G(P_\gamma)$ with $i_g \pi = \pi u$.
- The set $\mathcal{L}_{b_\sigma}(G_\sigma, L, u)$ is a $\big(G_\sigma, \mathrm{Aut}(L,u)\big)$-biset via the action defined by

$$g \cdot (P_\gamma, \pi) \cdot \varphi = ({}^g P_{{}^g\gamma}, i_g \pi \varphi)\,,$$

for $(g, \varphi) \in G_\sigma \times \mathrm{Aut}(L,u)$ and $(P_\gamma, \pi) \in \mathcal{L}_{b_\sigma}(G_\sigma, L, u)$. For $(P_\gamma, \pi) \in \mathcal{L}_{b_\sigma}(G_\sigma, L, u)$, we denote by $\overline{(P_\gamma, \pi)}$ the left orbit $G_\sigma(P_\gamma, \pi)$, and by $\mathrm{Aut}(L,u)_{\overline{(P_\gamma,\pi)}}$ the stabilizer of this orbit in $\mathrm{Aut}(L,u)$, namely

$$\mathrm{Aut}(L,u)_{\overline{(P_\gamma,\pi)}} = \{\varphi \in \mathrm{Aut}(L,u) \mid \exists g \in N_{G_\sigma}(P_\gamma),\ i_g\pi = \pi\varphi\}.$$

So we want to show that

$$\sum_{\sigma \in [G \backslash \mathcal{S}_p(G)]} (-1)^{|\sigma|} m_{L,u,V}(G_\sigma, b_\sigma) = 0$$

whenever $L \neq 1$. We observe that

$$\dim_{\mathbb{F}} V^{\mathrm{Aut}(L,u)_{\overline{(P_\gamma,\pi)}}} = \big(\mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P_\gamma,\pi)}}}^{\mathrm{Aut}(L,u)} \mathbb{F}, V\big)_{\mathrm{Aut}(L,u)}\,,$$

where $(-,-)_{\mathrm{Aut}(L,u)}$ denotes the Schur inner product on the Grothendieck group (character ring) $R_{\mathbb{F}}(\mathrm{Aut}(L,u))$ of finite dimensional $\mathbb{F}\mathrm{Aut}(L,u)$-modules. We set

$$W(G,L,u) = \sum_{\sigma \in [G \backslash \mathcal{S}_p(G)]} (-1)^{|\sigma|} \sum_{(P_\gamma,\pi) \in [G_\sigma \backslash \mathcal{L}_{b_\sigma}(G_\sigma, L, u)/\mathrm{Aut}(L,u)]} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P_\gamma,\pi)}}}^{\mathrm{Aut}(L,u)} \mathbb{F},$$

which we view as an element in $R_{\mathbb{F}}(\mathrm{Aut}(L,u))$. We want to show that

$$\big(W(G,L,u), V\big)_{\mathrm{Aut}(L,u)} = 0\,,$$

for all $L \neq 1$ and all simple $\mathbb{F}\mathrm{Aut}(L,u)$-modules V. But this amounts to saying that the virtual character $W(G,L,u)$ is equal to 0. Since $W(G,L,u)$ is a (virtual) permutation character its value at $\varphi \in \mathrm{Aut}(L,u)$ is equal to

$$|W(G,L,u)^\varphi| = \sum_{\sigma \in [G \backslash \mathcal{S}_p(G)]} (-1)^{|\sigma|} \big|\big(G_\sigma \backslash \mathcal{L}_{b_\sigma}(G_\sigma, L, u)\big)^\varphi\big|\,.$$

So, we want to show that this number is equal to 0 if $L \neq 1$.

Summing over all $\sigma \in \mathcal{S}_p(G)$ rather than representatives of G-orbits, we have that

$$\begin{aligned}
|W(G,L,u)^\varphi| &= \sum_{\sigma\in\mathcal{S}_p(G)} \frac{|G_\sigma|}{|G|}(-1)^{|\sigma|}\big|\big(G_\sigma\backslash\mathcal{L}_{b_\sigma}(G_\sigma,L,u)\big)^\varphi\big| \\
&= \sum_{\substack{\sigma\in\mathcal{S}_p(G)\\ (P_\gamma,\pi)\in\mathcal{L}_{b_\sigma}(G_\sigma,L,u)\\ G_\sigma(P_\gamma,\pi)\cdot\varphi=G_\sigma(P_\gamma,\pi)}} \frac{|G_\sigma|}{|G|}(-1)^{|\sigma|}\frac{|G_\sigma\cap G_{(P_\gamma,\pi)}|}{|G_\sigma|} \\
&= \sum_{\substack{\sigma\in\mathcal{S}_p(G)\\ (P_\gamma,\pi)\in\mathcal{L}_{b_\sigma}(G_\sigma,L,u)\\ G_\sigma(P_\gamma,\pi)\cdot\varphi=G_\sigma(P_\gamma,\pi)}} (-1)^{|\sigma|}\frac{|G_\sigma\cap G_{(P_\gamma,\pi)}|}{|G|}
\end{aligned}$$

where $G_{(P_\gamma,\pi)}$ is the left stabilizer of (P_γ,π), i.e.,

$$G_{(P_\gamma,\pi)} = \{g\in G \mid {}^gP=P,\ {}^g\gamma=\gamma,\ i_g\pi=\pi\} = C_G(P)\cap N_G(P_\gamma).$$

Now $G_\sigma(P_\gamma,\pi)\cdot\varphi = G_\sigma(P_\gamma,\pi)$ if and only if there exists $g\in G_\sigma$ such that $({}^gP_{{}^g\gamma}, i_g\pi) = (P_\gamma,\pi\varphi)$, and in this case, the number of such elements $g\in G_\sigma$ is equal to $|G_\sigma\cap G_{(P_\gamma,\pi)}|$. It follows that

$$|W(G,L,u)^\varphi| = \frac{1}{|G|}\sum_{\substack{(\sigma,P_\gamma,\pi,g)\\ \sigma\in\mathcal{S}_p(G)\\ P_\gamma\in\mathcal{L}_{b_\sigma}(G_\sigma,L,u)\\ g\in G_\sigma\\ ({}^gP_{{}^g\gamma},i_g\pi)=(P_\gamma,\pi\varphi)}} (-1)^{|\sigma|}\,.$$

We can rewrite this as

$$|W(G,L,u)^\varphi| = \frac{1}{|G|}\sum_{(\sigma,P,\gamma,\pi,g)\in\mathbb{S}}(-1)^{|\sigma|},$$

where $\mathbb{S}$ is the set of quintuples (σ,P,γ,π,g) such that:

- $\sigma\in\mathcal{S}_p(G)$,
- $P\leqslant G_\sigma$,
- γ is a local point of $(kG_\sigma b_\sigma)^P$,
- $\pi: L\overset{\cong}{\to}P$ is a group isomorphism such that $\pi u\pi^{-1}\in N_{G_\sigma}(P_\gamma)$,
- $g\in G_\sigma$ is such that $({}^gP_{{}^g\gamma}, i_g\pi) = (P_\gamma,\pi\varphi)$, in other words $g\in N_{G_\sigma}(P_\gamma)$ and $i_g\pi = \pi\varphi$.

The proof that $|W(G,L,u)^\varphi| = 0$ is inspired by the proof of Lemma 4.1 of [7]. We will build an involution $(\sigma, P, \gamma, \pi, g) \mapsto (\sigma', P, \gamma', \pi, g)$ of $\mathbb{S}$ such that $|\sigma'| = |\sigma| \pm 1$.

Let $(\sigma, P, \gamma, \pi, g) \in \mathbb{S}$, with $\sigma = (1 = P_0 < P_1 < \ldots < P_n)$. Since $P \cong L \neq 1 = P_0$, there is a largest integer $i \in \{0, \ldots, n\}$ such that $P \not\leqslant P_i$. There are two cases:

- Either $i = n$ or $PP_i < P_{i+1}$, then set $\sigma' = \sigma \sqcup \{PP_i\}$, i.e.,

  $$\sigma' = (P_0 < \ldots < P_n < PP_n) \quad \text{or} \quad \sigma' = (P_0 < P_1 < \ldots P_i < PP_i < P_{i+1} < \ldots < P_n)\,,$$

  respectively.

- Or $PP_i = P_{i+1}$, and then set $\sigma' = \sigma \smallsetminus \{P_{i+1}\}$, i.e.,

  $$\sigma' = (P_0 < P_1 < \ldots P_i < P_{i+2} < \ldots < P_n).$$

One checks easily that $(\sigma')' = \sigma$, and it is clear that $|\sigma'| = |\sigma| \pm 1$. Moreover $P \leqslant G_{\sigma'}$ if and only if $P \leqslant G_\sigma$, and $N_G(P) \cap G_\sigma = N_G(P) \cap G_{\sigma'}$, i.e., $N_{G_\sigma}(P) = N_{G_{\sigma'}}(P)$.

Now by [10, Corollary 37.6], the Brauer morphism $\mathrm{Br}_P^{G_\sigma}$ induces a bijection between the local points of $(kG_\sigma b_\sigma)^P$ and the points of $kC_{G_\sigma}(P)\mathrm{Br}_P^{G_\sigma}(b_\sigma)$. Since $N_{G_\sigma}(P) = N_{G_{\sigma'}}(P)$, we also have $C_{G_\sigma}(P) = C_{G_{\sigma'}}(P)$. Moreover, $\mathrm{Br}_P^{G_\sigma}(b_\sigma) = \mathrm{Br}_P^{G_{\sigma'}}(b_{\sigma'})$. In fact, $\mathrm{Br}_P^{G_\sigma}(b_\sigma)$ is the truncation to $kC_{G_\sigma}(P)$ of $b_\sigma \in kC_{G_\sigma}(P_n)$, so $\mathrm{Br}_P^{G_\sigma}(b_\sigma)$ is the truncation of b to $kC_{G_\sigma}(PP_n) = kC_{G_{\sigma'}}(PP_n)$, which is equal to $\mathrm{Br}_P^{G_{\sigma'}}(b_{\sigma'})$.

It follows that the Brauer morphism at P induces a bijection $\gamma \mapsto \gamma'$ between the local points of $(kG_\sigma b_\sigma)^P$ and those of $(kG_{\sigma'} b_{\sigma'})^P$. This bijection is $N_{G_\sigma}(P)$-equivariant, so $N_{G_\sigma}(P_\gamma) = N_{G_{\sigma'}}(P_{\gamma'})$, and it follows that $(\sigma', P, \gamma', \pi, g) \in \mathbb{S}$. Now $(\sigma, P, \gamma, \pi, g) \mapsto (\sigma', P, \gamma', \pi, g)$ is an involution of the set $\mathbb{S}$, with the property that $|\sigma'| = |\sigma| \pm 1$. Hence $|W(G,L,u)^\varphi| = 0$ for any $\varphi \in \mathrm{Aut}(L,u)$ if $L \neq \mathbf{1}$. This completes the proof of Assertion 1.

2. It follows from Assertion 1 that $\mathrm{FAwc}(G,b)$ and $\mathrm{Awc}(G,b)$ are both equivalent to $n_{G,b}$ being equal to 1 if $d(b) = 0$, and to 0 otherwise. Both parts of Assertion 2 follow. □

3.6 Corollary *Let (G,b) be a group-block pair over k. Then the following stable version of Conjecture FAwc(G,b) holds:*

$$\sum_{\sigma \in [G \backslash \mathcal{S}_p(G)]} (-1)^{|\sigma|} \, \overline{[\![G_\sigma, b_\sigma]\!]_{\mathbb{F}}} = 0 \text{ in } K_0(\overline{\mathcal{FB}\ell_{\mathbb{F},k}})\,.$$

Proof Indeed, the simple functor $S_{\mathbf{1},1,\mathbb{F}}$ becomes zero in the category $\overline{\mathcal{FB}\ell_{\mathbb{F},k}}$ and the result follows from Theorem 3.5. □

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