

The D^Δ -pair biset category

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Abstract

Let p be a prime number. A D^Δ -pair is a pair consisting of a finite p -group and a p' -automorphism of the group. In this paper, we introduce diagonal D^Δ -pair bisets and a category whose objects are D^Δ -pairs and whose morphism groups are Grothendieck groups of diagonal D^Δ -pair bisets. Our main result shows that, over suitable coefficient rings, the category of diagonal p -permutation functors is equivalent to the category of linear functors on a natural quotient of this new category. In this way, diagonal p -permutation functors can be studied through a category built only from finite p -groups and their automorphisms of p' -order.

MSC2020: 18A25, 19A22, 20C20, 20J15.

Keywords: bisets, diagonal p -permutation functors, D^Δ -pairs.

1 Introduction

Let p be a prime number, let R be a unital commutative ring, and k be an algebraically closed field of characteristic p . The diagonal p -permutation category Rpp_k^Δ ([5],[6]), has all the finite groups as objects, and morphisms given by R -linear extensions of Grothendieck groups of diagonal p -permutation bimodules. The category of diagonal p -permutation functors over R , that is, R -linear functors from Rpp_k^Δ to the category $R\mathbf{Mod}$ of all R -modules, was introduced in [5] (in the case that R is a field of characteristic 0), and developed in [6], [7] and [8]. It has allowed for interesting new results in block theory: To each *group-block* pair (G, b) , that is a pair of a finite group G and a block idempotent b of kG , is naturally attached a diagonal p -permutation functor $RT_{G,b}^\Delta$ over R . This leads to the notion of *functorial equivalence* (over R) of such group-block pairs ([6]).

When R is an algebraically closed field of characteristic zero, the category of diagonal p -permutation functors over R is semisimple. Its simple objects $S_{L,u,V}$ are parametrized by equivalence classes of triples (L, u, V) where (L, u) is a D^Δ -pair, that is, a pair consisting of a finite p -group L and a p' -automorphism u of L , and V is a simple $R\mathbf{Out}(L, u)$ -module. The multiplicity of a simple functor $S_{L,u,V}$ as a direct summand of $RT_{G,b}^\Delta$ now appears as a new invariant attached to a group block pair (G, b) and a triple (L, u, V) . These invariants yield in particular a finiteness theorem ([6]) in the spirit of Donovan's conjecture, saying that for a given p -group D , there is only a finite number of group block pairs (G, b) , where

b has defect groups isomorphic to D , up to functorial equivalence over R . More recently, this led us to an equivalent formulation of Alperin's blockwise weight conjecture in terms of diagonal p -permutation functors ([1]).

Our main goal of the present paper is the introduction of a new R -linear category $R\mathcal{P}^\Delta$, see Section 7. Its objects are the D^Δ -pairs and its morphisms are the scalar extensions to R of Grothendieck groups of so-called diagonal " D^Δ -pair bisets" (also newly introduced here, see Section 4). This category can be seen as an analogue of the biset category over R , where the objects are all finite groups and the morphisms come from bisets for finite groups. The category $R\mathcal{P}^\Delta$ has a particular quotient category $\overline{R\mathcal{P}^\Delta}$ with the same objects, and morphisms given by natural equivalence classes of morphisms in $R\mathcal{P}^\Delta$. This quotient category has the following remarkable property: when R is an integral domain in which all p' -integers are invertible and which contains all roots of unity of p' -order, then the category of R -linear functors from $\overline{R\mathcal{P}^\Delta}$ to $R\mathbf{Mod}$ is equivalent to the category of R -linear functors from Rpp_k^Δ to $R\mathbf{Mod}$, see Section 11. Thus, when considering R -linear functors, one can replace the more complicated theory of p -permutation kG -modules for arbitrary finite groups G with the theory of how D^Δ -pairs (L, u) act on finite sets. Our hope is that, using this equivalence, one can obtain new information on the structure of the diagonal p -permutation functors $RT_{G,b}^\Delta$ attached to group-block pairs, only from the knowledge of related information on p -groups and their automorphisms of p' -order.

1.1 Notation For any group G and $x \in G$, we denote by $i_x: G \rightarrow G$ the conjugation map $g \mapsto xgx^{-1}$. We also set ${}^xg := i_x(g)$ and $x^g := x^{-1}gx$. We denote by $|G|$ the order of G and will sometimes denote the order of a group element $g \in G$ by $|g|$.

2 D^Δ -pairs

In this section we define the category D^Δ -pair of D^Δ -pairs for a fixed prime p .

2.1 Definition A D^Δ -pair (L, u) is a pair consisting of a finite p -group L and an element $u \in \text{Aut}(L)$ of p' -order, i.e., of order not divisible by p . A *morphism* between two D^Δ -pairs (L, u) and (M, v) is a group homomorphism $f: L \rightarrow M$ satisfying $v \circ f = f \circ u: L \rightarrow M$. Together with the usual composition of group homomorphisms, this defines a category D^Δ -pair. In fact, if also $g: (M, v) \rightarrow (N, w)$ is a morphism of D^Δ -pairs, then the composition $g \circ f$ satisfies $w \circ g \circ f = g \circ v \circ f = g \circ f \circ u$ and is therefore again a morphism. The identity morphism of (L, u) is id_L .

2.2 Remark (a) Associating to any finite p -group L the D^Δ -pair (L, id_L) extends in an obvious way to a functor from the category of finite p -groups and group homomorphisms to the category of D^Δ -pair. This functor is fully faithful.

(b) The category D^Δ -pair has finite products. For D^Δ -pairs (L, u) and (M, v) we set $(L, u) \times (M, v) := (L \times M, u \times v)$. Together with the usual projection maps $L \times M \rightarrow L$ and $L \times M \rightarrow M$, this is a categorical product of (L, u) and (M, v) .

(c) For any D^Δ -pair one can consider the semidirect product $L\langle u \rangle = L \rtimes \langle u \rangle$ whose elements will be written as lu^i with $l \in L$ and $i \in \mathbb{Z}$. One has the commutation rule $u^i l = u^i(l)u^i$ in $L\langle u \rangle$. Note that L is the unique Sylow p -subgroup of $L\langle u \rangle$, hence the set of all p -elements of $L\langle u \rangle$.

2.3 Lemma *Let (L, u) and (M, v) be D^Δ -pairs and let $f : L \rightarrow M$ be a surjective group homomorphism satisfying $v \circ f = f \circ u$. Then there exists a unique extension $F : L\langle u \rangle \rightarrow M\langle v \rangle$ of f with $F(u) = v$.*

Proof The uniqueness of F is clear, since L and u generate $L\langle u \rangle$. To prove the existence, first note that if $i \in \mathbb{Z}$ is such that $u^i = \text{id}_L$ then also $v^i = \text{id}_M$. Indeed, $u^i = \text{id}_L$ implies $v^i \circ f = f \circ u^i = f$ and since f is surjective, this implies $v^i = \text{id}$. This now implies that the function $F : L\langle u \rangle \rightarrow M\langle v \rangle$, $lu^i \mapsto f(l)v^i$, where $l \in L$ and $i \in \mathbb{Z}$, is well-defined. The function F clearly extends f and it remains to be shown that F is a group homomorphism. So let $l, l' \in L$ and $i, j \in \mathbb{Z}$. Then

$$F(lu^i l' u^j) = F(lu^i(l')u^{i+j}) = f(lu^i(l'))v^{i+j}$$

and also

$$F(lu^i)F(l' u^j) = f(l)v^i f(l')v^j = f(l)v^i(f(l'))v^{i+j} = f(l)f(u^i(l'))v^{i+j} = f(lu^i(l'))v^{i+j}.$$

□

2.4 Corollary *Let (L, u) and (M, v) be D^Δ -pairs. Then the following are equivalent:*

- (i) $(L, u) \cong (M, v) \in D^\Delta$ -pair.
- (ii) There exists a group isomorphism $f : L \rightarrow M$ satisfying $v \circ f = f \circ u$.
- (iii) There exists a group isomorphism $F : L\langle u \rangle \rightarrow M\langle v \rangle$ satisfying $F(u) = v$.
- (iv) There exists a group isomorphism $F : L\langle u \rangle \rightarrow M\langle v \rangle$ such that $F(u)$ is $M\langle v \rangle$ -conjugate to v .

Proof (i) and (ii) are equivalent by definition, since if f is as in (ii) then also f^{-1} is a morphism of D^Δ -pairs. By Lemma 2.3, (ii) implies (iii). Moreover, (iii) obviously implies (iv). To prove that (iv) implies (ii), let $F : L\langle u \rangle \rightarrow M\langle v \rangle$ be a group isomorphism and $m \in M$ such that $F(u) = v^m$. Then $F|_L$ is a group isomorphism $L \rightarrow M$ between the respective unique Sylow p -subgroups of $L\langle u \rangle$ and $M\langle v \rangle$. Moreover, for any $l \in L$, the group isomorphism $f := i_m \circ F|_L : L \rightarrow M$ satisfies

$$\begin{aligned} (v \circ f)(l) &= v(i_m(F(l))) = v \cdot i_m(F(l)) \cdot v^{-1} = i_m(v^m \cdot F(l) \cdot (v^{-1})^m) \\ &= i_m(F(u) \cdot F(l) \cdot F(u^{-1})) = i_m(F(u \cdot l \cdot u^{-1})) = i_m(F(u(l))) = f(u(l)), \end{aligned}$$

so that f satisfies (ii). This completes the proof. □

2.5 Remark Let (L, u) be a D^Δ -pair.

(a) Suppose that $X \leq L$ is u -invariant, i.e., $u(X) = X$. Then the restriction $u' := u|_X$ of u to X is an automorphism of X of p' -order, so that (X, u') is a D^Δ -pair. In this case we call (X, u') a D^Δ -subpair of (L, u) . Note that the inclusion map $\iota : X \rightarrow L$ is a morphism from (X, u') to (L, u) in the category D^Δ -pair.

(b) Suppose that $X \trianglelefteq L$ is u -invariant. Then there exists a unique element $\bar{u} \in \text{Aut}(L/X)$ such that $\bar{u} \circ \pi = \pi \circ u$, where $\pi : L \rightarrow L/X$ denotes the canonical projection. In this case we call $(L/X, \bar{u})$ a *quotient D^Δ -pair* of (L, u) . Note that the map π is a morphism from (L, u) to $(L/X, \bar{u})$ in D^Δ -pair.

3 D^Δ -pairs acting on sets

In this section we define, for a D^Δ -pair (L, u) , the category ${}_{(L,u)}\mathbf{set}$ of (L, u) -sets. Recall that $L\langle u \rangle = L \rtimes \langle u \rangle$. If $u = \text{id}_L$, this specializes to the usual category of finite L -sets.

3.1 Definition Let (L, u) be a D^Δ -pair. An (L, u) -set is a finite L -set Ω equipped with an action of $\langle u \rangle$, denoted by $\omega \mapsto u \cdot \omega$ with the following properties:

(i) If $l \in L$ and $\omega \in \Omega$, then

$$u \cdot (l\omega) = u(l)(u \cdot \omega).$$

(ii) Every L -orbit on Ω is invariant by u , that is,

$$\forall \omega \in \Omega, u \cdot \omega \in L\omega.$$

3.2 Remark (a) Condition (i) is equivalent to saying that Ω is an $L\langle u \rangle$ -set. Thus, an (L, u) -set is nothing else than an $L\langle u \rangle$ -set with the property that its $L\langle u \rangle$ -orbits and L -orbits coincide. This implies that a disjoint union of finitely many $L\langle u \rangle$ -sets Ω_i , $i \in I$, is an (L, u) -set if and only if every Ω_i , $i \in I$, is an (L, u) -set. In particular, the $L\langle u \rangle$ -orbits of an (L, u) -set are again (L, u) -sets.

(b) If Ω is an (L, u) -set then, by Glauberman's Lemma ([9] Lemma 3.24), Condition (ii) implies that each L -orbit contains a point fixed by u and that the $\langle u \rangle$ -fixed points in an L -orbit form a single L^u -orbit. Here, we denote by L^u the set of u -fixed points on L , that is

$$L^u := \{l \in L \mid u(l) = l\}.$$

Note that $L^u = L \cap C_{L\langle u \rangle}(u)$. We will denote the set of u -fixed points of Ω by Ω^u . Thus, the sets of L^u -orbits of Ω^u and $L\langle u \rangle$ -orbits of Ω are in canonical bijection, by mapping the L^u -orbits of $\omega \in \Omega^u$ to its $L\langle u \rangle$ -orbit (or equivalently L -orbit).

3.3 Definition A *morphism* of (L, u) -sets is just a morphism of $L\langle u \rangle$ -sets. This way one obtains a category ${}_{(L,u)}\mathbf{set}$ of (L, u) -sets, which is the full subcategory ${}_{L\langle u \rangle}\mathbf{set}$ consisting of (L, u) -sets. By Part (a), disjoint unions together with their inclusions form coproducts in the category ${}_{(L,u)}\mathbf{set}$.

3.4 Definition An (L, u) -set Ω is called *transitive* if it cannot be written as a disjoint union of two non-empty (L, u) -sets. By Remark 3.2(a), this just means that Ω is transitive as $L\langle u \rangle$ -set.

The goal for the remainder of this section is to obtain a better understanding of transitive (L, u) -sets.

3.5 Lemma Let (L, u) be a D^Δ -pair and let $X \leq L$. Consider the ordinary transitive L -set L/X . One obtains a well-defined action of $\langle u \rangle$ on L/X by setting

$$u^i \cdot lX := u^i(l)X \quad (l \in L, i \in \mathbb{Z})$$

if and only if X is u -invariant. In this case this definition endows L/X with a transitive (L, u) -set structure.

Proof We first show that the action of $\langle u \rangle$ on L/X is well-defined if and only if X is u -invariant. Suppose first that the action of $\langle u \rangle$ is well-defined. Then for $x \in X$, we have

$$u(x)X = u \cdot xX = u \cdot X = X.$$

Thus $u(x) \in X$ and hence $u(X) = X$. Conversely, suppose that X is u -invariant. If $lX = l'X$, then $l^{-1}l' \in X$, and so $u^i(l^{-1})u^i(l') = u^i(l^{-1}l') \in X$ for any $i \in \mathbb{Z}$. Thus,

$$u^i(l)X = u^i(l')X,$$

so the action of $\langle u \rangle$ is well-defined.

Now, for $l_1, l_2 \in L$, we have

$$u \cdot (l_1 l_2 X) = u(l_1 l_2)X = u(l_1)u(l_2)X = u(l_1)(u \cdot l_2 X),$$

so the L -action and the $\langle u \rangle$ -action are compatible. Also, for any $l \in L$,

$$u \cdot (lX) = u(l)X \in L/X.$$

Therefore L/X is an (L, u) -set. Finally, L/X is clearly a transitive (L, u) -set. $\square$

3.6 Lemma Every transitive (L, u) -set is isomorphic to L/X (with its structure as in Lemma 3.5) for some u -invariant subgroup X of L .

Proof Let Ω be a transitive (L, u) -set. Then Ω is transitive as an L -set. By Glauberman's Lemma Ω contains a point ω fixed by u . Set

$$X = \text{Stab}_L(\omega).$$

Then X is u -invariant. Indeed, if $x \in X$, then

$$u(x)\omega = u(x)(u \cdot \omega) = u \cdot (x\omega) = u \cdot \omega = \omega.$$

Hence $u(x) \in X$.

Now, the usual L -set isomorphism

$$f : L/X \longrightarrow \Omega, \quad lX \mapsto l\omega$$

is also $\langle u \rangle$ -equivariant, since

$$f(u \cdot (lX)) = f(u(l)X) = u(l)f(X) = u(l)\omega = u(l)(u \cdot \omega) = u \cdot (l\omega) = u \cdot f(lX).$$

Hence $\Omega \cong L/X$ as (L, u) -sets. $\square$

3.7 Lemma *Let $X, Y \leq L$ be u -invariant subgroups. Then L/X and L/Y are isomorphic as (L, u) -sets if and only if there exists $a \in L^u$ such that*

$$X = aYa^{-1}.$$

Proof If $a \in L^u$ with $X = aYa^{-1}$, then the map

$$f : L/X \longrightarrow L/Y, \quad lX \mapsto laY$$

is an isomorphism of L -sets. It is also compatible with the action of u . Indeed, since $u(a) = a$, we have

$$f(u \cdot (lX)) = f(u(l)X) = u(l)aY = u(l)u(a)Y = u(la)Y = u \cdot laY = u \cdot f(lX).$$

Conversely, suppose that

$$\varphi : L/X \longrightarrow L/Y$$

is an isomorphism of (L, u) -sets. In particular, φ is an isomorphism of transitive L -sets. Hence there exists $b \in L$ such that

$$X = bYb^{-1} \quad \text{and} \quad \varphi(lX) = lbY.$$

Since φ is $\langle u \rangle$ -equivariant and the element $X \in L/X$ is u -fixed, it follows that $bY = \varphi(X)$ is also u -fixed. The element $Y \in L/Y$ is also u -fixed. By Glauberman's Lemma, the u -fixed elements of the transitive L -set L/Y form a transitive L^u -set. Hence there exists $a \in L^u$ such that

$$bY = aY.$$

Therefore

$$X = bYb^{-1} = aYa^{-1}.$$

This proves the claim. $\square$

3.8 Corollary *The isomorphism classes of transitive (L, u) -sets are in bijection with the L^u -conjugacy classes of u -invariant subgroups of L .*

Proof By Lemma 3.6, every transitive (L, u) -set is isomorphic to L/X for some u -invariant subgroup $X \leq L$, and by Lemma 3.7 two such transitive (L, u) -sets L/X and L/Y are isomorphic if and only if X and Y are conjugate by an element of L^u . $\square$

4 The $((M, v), (L, u))$ -bisets

4.1 Definition Let (L, u) and (M, v) be D^Δ -pairs. An $((M, v), (L, u))$ -biset is a finite (M, L) -biset Ω equipped with an action of $\langle(v, u)\rangle$, denoted by

$$\omega \mapsto (v, u) \cdot \omega,$$

with the following properties:

- (i) If $m \in M$, $l \in L$, and $\omega \in \Omega$, then

$$(v, u) \cdot (m\omega l) = v(m)((v, u) \cdot \omega)u(l).$$

- (ii) Every (M, L) -orbit on Ω is invariant under (v, u) , that is,

$$\forall \omega \in \Omega, \quad (v, u) \cdot \omega \in M\omega L.$$

4.2 Remark With the usual way of translating a right action to a left action, an $((M, v), (L, u))$ -biset is the same thing as an $(M \times L, (v, u))$ -set, as in Definition 3.1, where (v, u) denotes the automorphism of $M \times L$ given by $(m, l) \mapsto (v(m), u(l))$. Equivalently, an $((M, v), (L, u))$ -biset is an $(M \times L) \rtimes \langle(v, u)\rangle$ -set with the property that every $M \times L$ -orbit is invariant under (v, u) . We therefore have a category

$$(M, v)\mathbf{set}_{(L, u)} := (M \times L, (v, u))\mathbf{set}$$

of $((M, v), (L, u))$ -bisets, which can be considered as the full subcategory of $(M \times L)\langle(v, u)\rangle$ -sets whose objects satisfy that their $M \times L$ -orbits are invariant under (v, u) , or equivalently that their $(M \times L)$ -orbits and $(M \times L)\langle(v, u)\rangle$ -orbits coincide (see Remark 3.2(a)).

Glauberman's Lemma applies in the same way as in Remark 3.2(b) with $(M \times L)^{(v, u)} = M^v \times L^u$.

4.3 Remark Let $(\mathbf{1}, 1)$ denote the *trivial* D^Δ -pair, consisting of the trivial p -group $\mathbf{1}$, and its unique (identity) automorphism. Then, for any D^Δ -pair (L, u) , the $((L, u), (\mathbf{1}, 1))$ -bisets identify trivially with the (L, u) -sets. This identification is in fact an isomorphism of categories between $_{(L, u)}\mathbf{set}_{(\mathbf{1}, 1)}$ and $_{(L, u)}\mathbf{set}$.

The following Proposition follows immediately from Lemma 3.5, Lemma 3.7, and Corollary 3.8.

4.4 Proposition Let (L, u) and (M, v) be D^Δ -pairs. Up to isomorphism, the transitive $((M, v), (L, u))$ -bisets are precisely the transitive $(M \times L)\langle(v, u)\rangle$ -sets of the form $(M \times L)/X$, where $X \leq M \times L$ is (v, u) -invariant.

If X and Y are (v, u) -invariant subgroups of $M \times L$ then $(M \times L)/X$ and $(M \times L)/Y$ are isomorphic as $((M, v), (L, u))$ -bisets if and only if X and Y are conjugate by an element in $M^v \times L^u$. In particular, the isomorphism classes of transitive $((M, v), (L, u))$ -bisets are in bijection with the $(M^v \times L^u)$ -conjugacy classes of (v, u) -invariant subgroups of $M \times L$.

In the following definition we use the *composition* of two bisets. See [3, Definition 2.3.11] for the definition of this construction and related notations.

4.5 Definition Let (L, u) , (M, v) and (N, w) be D^Δ -pairs. Let Ω be an $((M, v), (L, u))$ -biset and let Γ be an $((N, w), (M, v))$ -biset. We denote by

$$\Omega^{(v, u)} = \{\omega \in \Omega \mid (v, u) \cdot \omega = \omega\}$$

the set of $\langle (v, u) \rangle$ -fixed points on Ω . Then $\Omega^{(v, u)}$ is naturally an (M^v, L^u) -biset. Indeed, for $(m, l) \in M^v \times L^u$ and $\omega \in \Omega^{(v, u)}$, one has

$$(v, u) \cdot (m\omega l) = v(m)((v, u) \cdot \omega)u(l) = m\omega l.$$

Similarly, $\Gamma^{(w, v)}$ is naturally an (N^w, M^v) -biset. Hence we may form the biset product

$$\Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)}.$$

There is a natural map

$$\theta: \Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)} \longrightarrow \Gamma \times_M \Omega, \quad (\gamma, M^v \omega) \mapsto (\gamma, M\omega).$$

We define the (N, L) -biset $\Gamma \times_{M, v} \Omega$ as the (N, L) -subbiset

$$\Gamma \times_{M, v} \Omega := N \cdot \text{Im}(\theta) \cdot L \subseteq \Gamma \times_M \Omega$$

of $\Gamma \times_M \Omega$. Equivalently, $\Gamma \times_{M, v} \Omega$ is the (N, L) -subbiset of $\Gamma \times_M \Omega$ generated by the elements $(\gamma, M\omega)$ with

$$\gamma \in \Gamma^{(w, v)} \quad \text{and} \quad \omega \in \Omega^{(v, u)}.$$

4.6 Lemma *With the notation from the previous definition, the (N, L) -biset $\Gamma \times_{M, v} \Omega$ is an $((N, w), (L, u))$ -biset with the $\langle (w, u) \rangle$ -action given by*

$$(w, u) \cdot (\gamma, M\omega) = ((w, v) \cdot \gamma, M(v, u) \cdot \omega),$$

for any element $(\gamma, M\omega) \in \Gamma \times_{M, v} \Omega$. Moreover, this defines a functor

$$(N, w) \mathbf{set}_{(M, v)} \times (M, v) \mathbf{set}_{(L, u)} \rightarrow (N, w) \mathbf{set}_{(L, u)}, \quad (\Gamma, \Omega) \mapsto \Gamma \times_{M, v} \Omega,$$

which preserves coproducts in both arguments.

Proof First we observe that the function

$$\sigma: \Gamma \times_M \Omega \rightarrow \Gamma \times_M \Omega, \quad (\gamma, M\omega) \mapsto ((w, v) \cdot \gamma, M(v, u) \cdot \omega),$$

is well defined. Indeed, for $m \in M$, one has

$$\begin{aligned} ((w, v) \cdot (\gamma m), M(v, u) \cdot (m^{-1}\omega)) &= (((w, v) \cdot \gamma)v(m), Mv(m^{-1})((v, u) \cdot \omega)) \\ &= ((w, v) \cdot \gamma, M(v, u) \cdot \omega). \end{aligned}$$

Moreover, if $\gamma \in \Gamma^{(w,v)}$ and $\omega \in \Omega^{(v,u)}$, then

$$\sigma(\gamma, {}_M\omega) = (\gamma, {}_M\omega).$$

Thus, every element of $\text{Im}(\theta)$ is fixed by σ . It follows that the (N, L) -subbiset generated by $\text{Im}(\theta)$ is stable under σ , because

$$\begin{aligned} \sigma(n\gamma, {}_M\omega l) &= ((w, v) \cdot (n\gamma), {}_M(v, u) \cdot (\omega l)) = (w(n)((w, v) \cdot \gamma), {}_M((v, u) \cdot \omega)u(l)) \\ &= (w(n)\gamma, {}_M\omega u(l)), \end{aligned} \quad (1)$$

for any $n \in N$, $l \in L$, $\gamma \in \Gamma^{(w,v)}$ and $\omega \in \Omega^{(v,u)}$. Therefore, σ restricts to a map $\Gamma \times_{M,v} \Omega \rightarrow \Gamma \times_{M,v} \Omega$. Moreover, if $(w, u)^i = 1$ then the last computation shows that $\sigma^i(\gamma, {}_M\omega) = (\gamma, {}_M\omega)$ for any $(\gamma, {}_M\omega) \in \Gamma \times_{M,v} \Omega$. Thus, setting

$$(w, u) \cdot (\gamma, {}_M\omega) := \sigma(\gamma, {}_M\omega) = ((w, v) \cdot \gamma, {}_M(v, u) \cdot \omega),$$

for $(\gamma, {}_M\omega) \in \Gamma \times_{M,v} \Omega$, defines an action of $\langle (w, u) \rangle$ on $\Gamma \times_{M,v} \Omega$. Equation (1) also shows that

$$(w, u) \cdot (n(\gamma, {}_M\omega)l) = w(n)(\gamma, {}_M\omega)u(l),$$

for all $(\gamma, {}_M\omega) \in \Gamma \times_{M,v} \Omega$ and $(n, l) \in N \times L$, so that $\Gamma \times_{M,v} \Omega$ becomes an $(N \times L) \rtimes \langle (w, u) \rangle$ -set. Clearly, every (N, L) -orbit in $\Gamma \times_{M,v} \Omega$ contains an element of $\text{Im}(\theta)$, hence contains an element fixed by (w, u) . Therefore, every (N, L) -orbit is invariant under (w, u) and $\Gamma \times_{M,v} \Omega$ is an $((N, w), (L, u))$ -biset.

If

$$\beta: \Gamma \rightarrow \Gamma'$$

is a morphism of $((N, w), (M, v))$ -bisets and

$$\alpha: \Omega \rightarrow \Omega'$$

is a morphism of $((M, v), (L, u))$ -bisets, then the induced map

$$\beta \times_M \alpha: \Gamma \times_M \Omega \longrightarrow \Gamma' \times_M \Omega', \quad (\gamma, {}_M\omega) \mapsto (\beta(\gamma), {}_M\alpha(\omega)),$$

restricts to a map

$$\Gamma \times_{M,v} \Omega \longrightarrow \Gamma' \times_{M,v} \Omega'.$$

Indeed, β sends $\Gamma^{(w,v)}$ into $(\Gamma')^{(w,v)}$, and α sends $\Omega^{(v,u)}$ into $(\Omega')^{(v,u)}$. Thus the image of the generating subset of $\Gamma \times_{M,v} \Omega$ is contained in the generating subset of $\Gamma' \times_{M,v} \Omega'$. Hence composition is functorial in both variables. Finally, it is straightforward to show that this construction preserves coproducts in both arguments. $\square$

We call the above $((N, w), (L, u))$ -biset $\Gamma \times_{M,v} \Omega$ the *composition* of the $((N, w), (M, v))$ -biset Γ and the $((M, v), (L, u))$ -biset Ω .

4.7 Proposition *Let (K, t) , (N, w) , (M, v) and (L, u) be D^Δ -pairs. Let Θ be a $((K, t), (N, w))$ -biset, let Γ be a $((N, w), (M, v))$ -biset, and let Ω be a $((M, v), (L, u))$ -biset. Then there is a natural isomorphism of $((K, t), (L, u))$ -bisets*

$$(\Theta \times_{N,w} \Gamma) \times_{M,v} \Omega \cong \Theta \times_{N,w} (\Gamma \times_{M,v} \Omega).$$

Proof We claim that the natural associativity isomorphism

$$(\Theta \times_N \Gamma) \times_M \Omega \cong \Theta \times_N (\Gamma \times_M \Omega), \quad (\theta, {}_N\gamma), {}_M\omega \mapsto (\theta, {}_N(\gamma, {}_M\omega)),$$

restricts to an isomorphism between

$$(\Theta \times_{N,w} \Gamma) \times_{M,v} \Omega \quad \text{and} \quad \Theta \times_{N,w} (\Gamma \times_{M,v} \Omega)$$

which is also $\langle (t, u) \rangle$ -equivariant. This will prove the result.

Let $S_{\Theta\Gamma}$ denote the image of

$$\Theta^{(t,w)} \times_{N^w} \Gamma^{(w,v)} \longrightarrow \Theta \times_N \Gamma.$$

By definition,

$$\Theta \times_{N,w} \Gamma = K S_{\Theta\Gamma} M.$$

We need to understand the (t, v) -fixed points of this composite. Since $\Theta \times_{N,w} \Gamma$ is generated, as a (K, M) -biset, by the (t, v) -fixed set $S_{\Theta\Gamma}$, every (K, M) -orbit in $\Theta \times_{N,w} \Gamma$ contains a (t, v) -fixed point. By Glauberman's Lemma, the fixed points in such an orbit form a single orbit under $K^t \times M^v$. Hence,

$$(\Theta \times_{N,w} \Gamma)^{(t,v)} = K^t S_{\Theta\Gamma} M^v.$$

Therefore

$$(\Theta \times_{N,w} \Gamma) \times_{M,v} \Omega$$

is the (K, L) -subbiset of $(\Theta \times_N \Gamma) \times_M \Omega$ generated by the image of

$$(K^t S_{\Theta\Gamma} M^v) \times_{M^v} \Omega^{(v,u)}.$$

This subbiset is generated by the elements $((\theta, {}_N\gamma), {}_M\omega)$ with

$$\theta \in \Theta^{(t,w)}, \quad \gamma \in \Gamma^{(w,v)}, \quad \omega \in \Omega^{(v,u)}.$$

The same argument on the other side gives

$$(\Gamma \times_{M,v} \Omega)^{(w,u)} = N^w S_{\Gamma\Omega} L^u,$$

where $S_{\Gamma\Omega}$ is the image of

$$\Gamma^{(w,v)} \times_{M^v} \Omega^{(v,u)} \longrightarrow \Gamma \times_M \Omega.$$

Hence,

$$\Theta \times_{N,w} (\Gamma \times_{M,v} \Omega)$$

is also the (K, L) -subbiset of $\Theta \times_N (\Gamma \times_M \Omega)$ generated by the elements $(\theta, {}_N(\gamma, {}_M\omega))$ with θ , γ , and ω fixed by (t, w) , (w, v) , and (v, u) respectively.

Thus, the usual associativity isomorphism of biset compositions restricts to an isomorphism

$$(\Theta \times_{N,w} \Gamma) \times_{M,v} \Omega \cong \Theta \times_{N,w} (\Gamma \times_{M,v} \Omega),$$

and this isomorphism is compatible with the action of $\langle(t, u)\rangle$. Therefore it is an isomorphism of $((K, t), (L, u))$ -bisets. $\square$

4.8 Proposition *For each D^Δ -pair (L, u) , the regular (L, L) -biset L , equipped with the action of $\langle(u, u)\rangle$ given by*

$$(u^i, u^i) \cdot l = u^i(l),$$

is an identity element for the composition of D^Δ -pair bisets.

Proof Let

$$\text{Id}_{(L,u)} := L$$

with its usual regular (L, L) -biset structure and with the action of $\langle(u, u)\rangle$ given by

$$(u^i, u^i) \cdot l = u^i(l).$$

The unique (L, L) -orbit is clearly invariant under (u, u) , so $\text{Id}_{(L,u)}$ is an $((L, u), (L, u))$ -biset.

We now show that $\text{Id}_{L,u}$ acts as a right identity. Let Ω be a $((M, v), (L, u))$ -biset. By definition, $\Omega \times_{L,u} \text{Id}_{(L,u)}$ is the (M, L) -subbiset of $\Omega \times_L L$ generated by the image of the map $\theta : \Omega^{(v,u)} \times_{L^u} \text{Id}_L^{(u,u)} \rightarrow \Omega \times_L L$ sending $(\omega, {}_{L^u}l)$ to $(\omega, {}_Ll)$, for $\omega \in \Omega$ and $l \in \text{Id}_L^{(u,u)} = L^u$. The map

$$\Omega^{(v,u)} \times_{L^u} L^u \longrightarrow \Omega, \quad (\omega, {}_{L^u}l) \longmapsto \omega l,$$

identifies $\Omega^{(v,u)} \times_{L^u} L^u$ with $\Omega^{(v,u)}$. Therefore

$$X \times_{L,u} \text{Id}_{(L,u)} = M\Omega^{(v,u)}L.$$

Since X is an $((M, v), (L, u))$ -biset, every $(M \times L)$ -orbit of Ω contains a point fixed by (v, u) , by Glauberman's Lemma. Hence

$$M\Omega^{(v,u)}L = \Omega.$$

This prove the claim. One can similarly show that $\text{Id}_{(L,u)}$ is also a left identity. $\square$

5 Functors associated to $((M, v), (L, u))$ -bisets

Let (M, v) and (L, u) be D^Δ -pairs, and Ω be an $((M, v), (L, u))$ -biset. By Remark 4.3, if Φ is an (L, u) -set, we can view Φ as an $((L, u), (\mathbf{1}, 1))$ -biset. Then $\Omega \times_{L, u} \Phi$ is an $((M, v), (\mathbf{1}, 1))$ -biset, hence an (M, v) -set. By Lemma 4.6, this construction $\Phi \mapsto \Omega \times_{L, u} \Phi$ is in fact a functor F_Ω from $_{(L, u)}\mathbf{set}$ to $_{(M, v)}\mathbf{set}$. We note that this functor preserves coproducts.

Moreover, if (N, w) is a D^Δ -pair, and if Γ is a $((N, w), (M, v))$ -biset, then Proposition 4.7 shows that the functors $F_\Gamma \circ F_\Omega$ and $F_{\Gamma \times_{M, v} \Omega}$, from $_{(L, u)}\mathbf{set}$ to $_{(N, w)}\mathbf{set}$, are isomorphic.

We consider five special cases of these functors F_Ω , associated to what we call *elementary* D^Δ -pair bisets, defined as follows:

5.1 Definition Let (L, u) be a D^Δ -pair, let $X \leq L$ be u -stable, and set $u' := u|_X \in \text{Aut}(X)$. Let also $Y \trianglelefteq L$ be u -stable, and let $\bar{u} \in \text{Aut}(L/Y)$ be the automorphism induced by u .

(a) The *restriction* biset

$$\text{Res}_{(X, u')}^{(L, u)} \in {}_{(X, u')}\mathbf{set}_{(L, u)}$$

is defined as the ordinary (X, L) -biset L , equipped with the action of $\langle(u', u)\rangle$ given by

$$(u'^i, u^i) \cdot l = u^i(l) \quad (l \in L, i \in \mathbb{Z}).$$

The corresponding functor from $_{(L, u)}\mathbf{set}$ to $_{(X, u')}\mathbf{set}$ will also be called *restriction*, and denoted by $\text{Res}_{(X, u')}^{(L, u)}$.

(b) The *induction* biset

$$\text{Ind}_{(X, u')}^{(L, u)} \in {}_{(L, u)}\mathbf{set}_{(X, u')}$$

is defined as the ordinary (L, X) -biset L , equipped with the action of $\langle(u, u')\rangle$ given by

$$(u^i, u'^i) \cdot l = u^i(l) \quad (l \in L, i \in \mathbb{Z}).$$

The corresponding functor from $_{(X, u')}\mathbf{set}$ to $_{(L, u)}\mathbf{set}$ will also be called *induction*, and denoted by $\text{Ind}_{(X, u')}^{(L, u)}$.

(c) The *inflation* biset

$$\text{Inf}_{(L/Y, \bar{u})}^{(L, u)} \in {}_{(L, u)}\mathbf{set}_{(L/Y, \bar{u})}$$

is defined as the ordinary $(L, L/Y)$ -biset L/Y , where L acts on L/Y through the quotient map $L \rightarrow L/Y$. We equip it with the action of $\langle(u, \bar{u})\rangle$ given by

$$(u^i, \bar{u}^i) \cdot (lY) = u^i(l)Y.$$

The corresponding functor from $(L/Y, \bar{u})\mathbf{set}$ to $(L, u)\mathbf{set}$ will also be called *inflation*, and denoted by $\mathrm{Inf}_{(L/Y, \bar{u})}^{(L, u)}$.

(d) The *deflation* biset

$$\mathrm{Def}_{(L/Y, \bar{u})}^{(L, u)} \in (L/Y, \bar{u})\mathbf{set}_{(L, u)}$$

is defined as the ordinary $(L/Y, L)$ -biset L/Y , where L acts on the right through the quotient map $L \rightarrow L/Y$. We equip it with the action of $\langle(\bar{u}, u)\rangle$ given by

$$(\bar{u}^i, u^i) \cdot (lY) = u^i(l)Y.$$

The corresponding functor from $(L, u)\mathbf{set}$ to $(L/Y, \bar{u})\mathbf{set}$ will also be called *deflation*, and denoted by $\mathrm{Def}_{(L/Y, \bar{u})}^{(L, u)}$.

(e) Let (M, v) be another D^Δ -pair and let $f : (L, u) \rightarrow (M, v)$ be an isomorphism of D^Δ -pairs. We define the *isomorphism* biset

$$\mathrm{Iso}_f \in (M, v)\mathbf{set}_{(L, u)}$$

to be the ordinary (M, L) -biset M , where M acts on the left by left multiplication and L acts on the right via f . We equip it with the action of $\langle(v, u)\rangle$ given by

$$(v^i, u^i) \cdot m = v^i(m) \quad (m \in M, i \in \mathbb{Z}).$$

The corresponding functor from $(L, u)\mathbf{set}$ to $(M, v)\mathbf{set}$ will also be called *isomorphism*, and denoted by Iso_f .

6 The diagonal D^Δ -pair bisets

6.1 Definition Let (L, u) and (M, v) be D^Δ -pairs. A *diagonal* $((M, v), (L, u))$ -biset Ω is an $((M, v), (L, u))$ -biset which is *bifree* as an (M, L) -biset, i.e., it is free as left M -set and as right L -set. A morphism of diagonal $((M, v), (L, u))$ -bisets is just a morphism of $(M \times L) \rtimes \langle(v, u)\rangle$ -sets. We denote the resulting category by

$$(M, v)\mathbf{set}_{(L, u)}^\Delta.$$

It is the full subcategory of $(M, v)\mathbf{set}_{(L, u)}$ consisting of diagonal $((M, v), (L, u))$ -bisets.

6.2 Remark There is an obvious notion of disjoint union of diagonal $((M, v), (L, u))$ -bisets. An $((M, v), (L, u))$ -biset X is diagonal if and only if each $(M \times L)\langle(v, u)\rangle$ -orbit of X is diagonal. A transitive diagonal $((M, v), (L, u))$ -biset is just a diagonal $((M, v), (L, u))$ -biset which is transitive as $(M \times L)\langle(v, u)\rangle$ -set. As in Remark 4.2, Glauberman's theorem applies: Every $M \times L$ -orbit of a diagonal $((M, v), (L, u))$ -biset X has a point fixed by (v, u) and the set of (v, u) -fixed points in a given $M \times L$ -orbit form an $M^v \times L^u$ -orbit.

6.3 Proposition *Let (L, u) and (M, v) be D^Δ -pairs. Up to isomorphism, the transitive diagonal $((M, v), (L, u))$ -bisets are precisely the transitive $(M \times L) \langle (v, u) \rangle$ -sets of the form $(M \times L)/X$, where $X \leq M \times L$ is a diagonal (v, u) -invariant subgroup, i.e.,*

$$X = \Delta(B, \alpha, A) := \{\alpha(a), a \mid a \in A\},$$

for an isomorphism $\alpha: A \rightarrow B$ between subgroups $A \leq L$ and $B \leq M$.

If X and Y are (v, u) -invariant diagonal subgroups of $M \times L$, then $(M \times L)/X$ and $(M \times L)/Y$ are isomorphic as diagonal $((M, v), (L, u))$ -bisets if and only if X and Y are conjugate by an element in $M^v \times L^u$. In particular, the isomorphism classes of transitive $((M, v), (L, u))$ -bisets are in bijection with the $(M^v \times L^u)$ -conjugacy classes of (v, u) -invariant diagonal subgroups of $M \times L$.

Proof This follows from Proposition 4.4, since an ordinary transitive (M, L) -biset is bifree if and only if its stabilizer in $M \times L$ is diagonal. $\square$

6.4 Lemma *Let (L, u) , (M, v) and (N, w) be D^Δ -pairs, let Ω be a diagonal $((M, v), (L, u))$ -biset and let Γ be a diagonal $((N, w), (M, v))$ -biset. Then the map*

$$(\gamma, {}_{M^v}\omega) \in \Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)} \longmapsto (\gamma, {}_M\omega) \in \Gamma \times_M \Omega$$

is injective.

Proof Suppose that $\gamma, \gamma' \in \Gamma^{(w, v)}$ and $\omega, \omega' \in \Omega^{(v, u)}$ are such that

$$(\gamma, {}_M\omega) = (\gamma', {}_M\omega').$$

Then there exists $\mu \in M$ such that

$$\gamma' = \gamma\mu^{-1} \quad \text{and} \quad \omega' = \mu\omega.$$

Taking the image under (w, v) of the first equality gives

$$\gamma' = \gamma\mu^{-1} = \gamma v(\mu)^{-1},$$

hence $v(\mu) = \mu$, since Γ is right free. In other words $\mu \in M^v$. Then

$$(\gamma', {}_{M^v}\omega') = (\gamma\mu^{-1}, {}_{M^v}\mu\omega) = (\gamma, {}_{M^v}\omega),$$

as was to be shown. $\square$

In what follows, we will consider

$$\Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)}$$

as a subset of

$$\Gamma \times_M \Omega$$

under the map θ from Definition 4.5.

6.5 Lemma *Let (L, u) , (M, v) and (N, w) be D^Δ -pairs, let Ω be a diagonal $((M, v), (L, u))$ -biset and let Γ be a diagonal $((N, w), (M, v))$ -biset. Then the composition $\Gamma \times_{M, v} \Omega$ is a diagonal $((N, w), (M, v))$ -biset.*

Proof All we have to do is to show that the (N, L) -biset $\Gamma \times_{M, v} \Omega$ is bifree. This follows from Lemma 6.4, since $\Gamma \times_{M, v} \Omega$ is a subbiset of $\Gamma \times_M \Omega$ and since $\Gamma \times_M \Omega$ is again bifree as (N, L) -biset. $\square$

Moreover we have the following:

6.6 Lemma *Let (L, u) , (M, v) and (N, w) be D^Δ -pairs, let Ω be a diagonal $((M, v), (L, u))$ -biset and let Γ be a diagonal $((N, w), (M, v))$ -biset. Then*

$$(\Gamma \times_{M, v} \Omega)^{(w, u)} = \Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)}.$$

Proof By the definition of the action of $\langle (w, u) \rangle$ on $\Gamma \times_{M, v} \Omega$, every element of

$$\Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)}$$

is fixed by (w, u) . Hence

$$\Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)} \subseteq (\Gamma \times_{M, v} \Omega)^{(w, u)}.$$

Conversely, let

$$z \in (\Gamma \times_{M, v} \Omega)^{(w, u)}.$$

By the definition of the composition, the element z lies in the (N, L) -orbit of some element

$$s = (\gamma, M^v \omega) \in \Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)}.$$

In particular, s is fixed by (w, u) , and the (N, L) -orbit NsL is stable under (w, u) . Since NsL contains the fixed point s , Glauberman's Lemma implies that $(NsL)^{(w, u)}$ is transitive under $N^w \times L^u$. Therefore, there exist $n \in N^w$ and $l \in L^u$ such that

$$z = nsl = n(\gamma, M^v \omega)l = (n\gamma, M^v \omega l).$$

Since $n \in N^w$, $l \in L^u$, $\gamma \in \Gamma^{(w, v)}$ and $\omega \in \Omega^{(v, u)}$, we have

$$(w, v) \cdot (n\gamma) = w(n)((w, v) \cdot \gamma) = n\gamma$$

and

$$(v, u) \cdot (\omega l) = ((v, u) \cdot \omega)u(l) = \omega l.$$

Therefore

$$n\gamma \in \Gamma^{(w, v)} \quad \text{and} \quad \omega l \in \Omega^{(v, u)}.$$

Hence

$$z = (n\gamma, M^v \omega l) \in \Gamma^{(w, v)} \times_{M^v} \Omega^{(v, u)}.$$

This proves the reverse inclusion, and hence the equality. $\square$

6.7 Proposition *Let (K, t) , (N, w) , (M, v) and (L, u) be D^Δ -pairs. Let Θ be a diagonal $((K, t), (N, w))$ -biset, let Γ be a diagonal $((N, w), (M, v))$ -biset, and let Ω be a diagonal $((M, v), (L, u))$ -biset. Then there is a natural isomorphism of diagonal $((K, t), (L, u))$ -bisets*

$$(\Theta \times_{N, w} \Gamma) \times_{M, v} \Omega \cong \Theta \times_{N, w} (\Gamma \times_{M, v} \Omega).$$

Proof This follows from Proposition 4.7 and Lemma 6.5. $\square$

6.8 Proposition *For each D^Δ -pair (L, u) , the regular (L, L) -biset L , equipped with the action of $\langle (u, u) \rangle$ given by*

$$(u^i, u^i) \cdot l = u^i(l),$$

is a diagonal $((L, u), (L, u))$ -biset, hence an identity element for the composition of diagonal D^Δ -pair bisets.

Proof By Proposition 4.8, all we have to show is that $\text{Id}_{(L, u)}$ is a diagonal $((L, u), (L, u))$ -biset. This follows from the fact that the underlying (L, L) -biset L is bifree. $\square$

Next we prove a formula for the composition of transitive diagonal D^Δ -pair bisets.

6.9 Proposition *Let (L, u) , (M, v) and (N, w) be D^Δ -pairs. Let*

$$\Gamma = (N \times M) / \Delta(T, \beta, Z)$$

be a transitive diagonal $((N, w), (M, v))$ -biset, and let

$$\Omega = (M \times L) / \Delta(Y, \alpha, X)$$

be a transitive diagonal $((M, v), (L, u))$ -biset, where $\Delta(T, \beta, Z)$ is (w, v) -invariant and $\Delta(Y, \alpha, X)$ is (v, u) -invariant (see Proposition 6.3). Then

$$\Gamma \times_{M, v} \Omega \cong \coprod_{m \in Z^v \setminus M^v / Y^v} (N \times L) / \Delta_m,$$

where

$$\Delta_m = \Delta(\beta(Z \cap {}^m Y), \beta i_m \alpha, \alpha^{-1}(Z^m \cap Y)).$$

Proof We abbreviate $\Delta_\beta := \Delta(T, \beta, Z)$ and $\Delta_\alpha := \Delta(Y, \alpha, X)$. First note that the element

$$\Delta_\beta := \Delta(T, \beta, Z) \in \Gamma = (N \times M) / \Delta(T, \beta, Z)$$

is fixed by (w, v) . By Glauberman's Lemma, we have

$$\Gamma^{(w, v)} = N^w \Delta_\beta M^v.$$

The stabilizer of the element Δ_β in $N^w \times M^v$ is

$$\Delta_\beta \cap (N^w \times M^v).$$

Since Δ_β is (w, v) -invariant, the isomorphism $\beta: Z \rightarrow T$ restricts to an isomorphism

$$\beta: Z^v \longrightarrow T^w.$$

Therefore

$$\text{stab}_{N^w \times M^v}(\Delta_\beta) = \Delta_\beta \cap (N^w \times M^v) = \Delta(T^w, \beta, Z^v).$$

Similarly,

$$\text{stab}_{M^v \times L^u}(\Delta_\alpha) = \Delta_\alpha \cap (M^v \times L^u) = \Delta(Y^v, \alpha, X^u).$$

By Lemma 6.5, $\Gamma \times_{M,v} \Omega$ is a diagonal $((N, w), (L, u))$ -biset and therefore an $(N \times L, (w, u))$ -set (see Remark 4.2). By Remark 3.2, the $N \times L$ -orbits of $\Gamma \times_{M,v} \Omega$ are in bijection with the $N^w \times L^u$ -orbits of $(\Gamma \times_{M,v} \Omega)^{(w,u)} = \Gamma^{(w,v)} \times_{M^v} \Omega^{(v,u)} \subseteq \Gamma \times_M \Omega$, see Lemma 6.6. Moreover, by [3, Lemma 2.3.24] and its proof, the $N^w \times L^u$ -orbits of $\Gamma^{(w,v)} \times_{M^v} \Omega^{(v,u)}$ are represented by the elements

$$(\Delta_{\beta, M^v} m \Delta_\alpha),$$

where $m \in M^v$ runs through a set of representatives of the the double cosets

$$p_2(\text{stab}_{N^w \times M^v}(\Delta_\beta)) \backslash M^v / p_1(\text{stab}_{M^v \times L^u}(\Delta_\alpha)) = Z^v \backslash M^v / Y^v.$$

Thus, altogether, the $N \times L$ -orbits of $\Gamma \times_{M,v} \Omega$ are represented by the elements

$$(\Delta_{\beta, M} m \Delta_\alpha) \in \Gamma \times_{M,v} \Omega.$$

But

$$\begin{aligned} \text{stab}_{N \times L}(\Delta_{\beta, M} m \Delta_\alpha) &= \text{stab}_{N \times M}(\Delta_\beta) * \text{stab}_{M \times L}(m \Delta_\alpha) \\ &= \text{stab}_{N \times M}(\Delta_\beta) * {}^{(m,1)}\Delta_\alpha = \Delta_m. \end{aligned}$$

This completes the proof. □

7 D^Δ -pair biset functors

7.1 Notation (a) Let (L, u) be a D^Δ -pair. We denote by

$$\mathcal{S}(L, u) = \{X \leq L \mid u(X) = X\}$$

the set of u -invariant subgroups of L , and by $[\mathcal{S}(L, u)]$ a set of representatives for the L^u -conjugacy classes in $\mathcal{S}(L, u)$.

We denote by $B(L, u)$ the Grothendieck group of finite (L, u) -sets with respect to disjoint union. By the classification of transitive (L, u) -sets (Corollary 3.8), $B(L, u)$ is the free abelian group with the set of isomorphism classes of transitive (L, u) -sets as basis. Equivalently,

$$B(L, u) = \bigoplus_{X \in [S(L, u)]} \mathbb{Z}[L/X].$$

(b) We similarly define the corresponding Burnside groups of D^Δ -pair bisets. Let (L, u) and (M, v) be D^Δ -pairs. We denote by

$$B((M, v), (L, u))$$

the Grothendieck group of finite $((M, v), (L, u))$ -biset; and by

$$B^\Delta((M, v), (L, u))$$

the Grothendieck group of finite diagonal $((M, v), (L, u))$ -bisets with respect to disjoint union.

For a commutative ring R , we set

$$RB(L, u) = R \otimes_{\mathbb{Z}} B(L, u).$$

Similarly we define $RB((M, v), (L, u))$ and $RB^\Delta((M, v), (L, u))$.

7.2 Definition Let R be a commutative ring. We define the R -linear D^Δ -pair biset category $R\mathcal{P}$ as follows.

- The objects are D^Δ -pairs.
- For two D^Δ -pairs (L, u) and (M, v) ,

$$\mathrm{Hom}_{R\mathcal{P}}((L, u), (M, v)) = RB((M, v), (L, u)).$$

- Composition is induced by the composition of D^Δ -pair bisets.
- The identity morphism of (L, u) is $[\mathrm{Id}_{(L, u)}]$.

Similarly, we define the R -linear diagonal D^Δ -pair biset category $R\mathcal{P}^\Delta$.

7.3 Definition Let R be a commutative ring. The R -linear diagonal D^Δ -pair biset category $R\mathcal{P}^\Delta$ is defined as follows:

- The objects are D^Δ -pairs.
- For two D^Δ -pairs (L, u) and (M, v) ,

$$\mathrm{Hom}_{R\mathcal{P}^\Delta}((L, u), (M, v)) = RB^\Delta((M, v), (L, u)).$$

- Composition is induced by the composition of D^Δ -pair bisets.
- The identity morphism of (L, u) is $[\text{Id}_{(L, u)}]$.

7.4 Definition (a) An R -linear D^Δ -pair biset functor is an R -linear functor

$$R\mathcal{P} \longrightarrow {}_R\mathbf{Mod}.$$

(b) An R -linear diagonal D^Δ -pair biset functor is an R -linear functor

$$R\mathcal{P}^\Delta \longrightarrow {}_R\mathbf{Mod}.$$

8 An equivalence of categories

Recall that p has been a fixed prime throughout this paper. For the remainder of this paper, we further fix an algebraically closed field k of characteristic p and an integral domain R with the property that every p' -integer is invertible in R and that R contains roots of unity of arbitrary p' -order. This implies that R has characteristic p or zero. In addition, we fix an imbedding $\iota: \mu(k) \rightarrow \mu(R)$ of the group $\mu(k)$ of roots of unity of k into the group $\mu(R)$ of roots of unity of R . Such an embedding always exists.

Next we recall some definitions from [5]. For finite groups G and H we denote by $T^\Delta(H, G)$ the Grothendieck group (with respect to direct sums) of p -permutation (kH, kG) -bimodules which are projective from either side, or equivalently, if considered as $k[H \times G]$ -modules, whose indecomposable direct summands have diagonal vertices in $H \times G$. Further, we set $RT^\Delta(H, G) := R \otimes_{\mathbb{Z}} T^\Delta(H, G)$. The diagonal p -permutation category Rpp_k^Δ has as objects all finite groups and as morphisms from G to H the set $RT^\Delta(H, G)$. Composition is induced by the tensor product of bimodules and the identity of G is the bimodule kG . The category of R -linear functors from Rpp_k^Δ to ${}_R\mathbf{Mod}$ is denoted by $\mathcal{F}_{Rpp_k}^\Delta$. It is an abelian R -linear category.

8.1 Notation (a) Let G be a finite group. We set $G^\natural := \text{Hom}(G, k^\times)$. The embedding $\iota: \mu(k) \rightarrow \mu(R)$ induces an embedding

$$G^\natural \rightarrow \text{Hom}(G, R^\times), \lambda \mapsto \tilde{\lambda}.$$

For $\lambda \in G^\natural$, we denote by k_λ the corresponding one-dimensional kG -module and by kG_λ the (kG, kG) -bimodule which is equal to kG as a k -vector space and with the action

$$g \cdot x \cdot h = \lambda(h)^{-1}gxh, \quad \text{for } g, x, h \in G.$$

(b) Let (L, u) be a D^Δ -pair and set $G := L\langle u \rangle$. For $x \in \langle u \rangle$, we set

$$E_{L, u}^G(x) = \frac{1}{|u|} \sum_{\lambda \in \langle u \rangle^\natural} \tilde{\lambda}(x)^{-1} kG_{\text{Id}, \lambda} \in RT^\Delta(G, G)$$

and

$$E_{L,u}^G := E_{L,u}^G(u).$$

8.2 Lemma *Let (L, u) be a D^Δ -pair and set $G := L\langle u \rangle$. The elements $E_{L,u}^G(x)$, for $x \in \langle u \rangle$ are pairwise orthogonal idempotents of $RT^\Delta(G, G)$ and their sum is equal to the identity element kG . In particular, the representable functor $RT^\Delta(-, G)$ splits as a direct sum*

$$\bigoplus_{x \in \langle u \rangle} RT^\Delta(-, G)E_{L,u}^G(x).$$

Proof For $x, y \in \langle u \rangle$, one has

$$\begin{aligned} E_{L,u}^G(x) \cdot E_{L,u}^G(y) &= \frac{1}{|u|^2} \sum_{\lambda, \mu \in \langle u \rangle^\natural} \tilde{\lambda}(x)^{-1} \tilde{\mu}(y)^{-1} kG_\lambda kG_\mu \\ &= \frac{1}{|u|^2} \sum_{\lambda, \mu \in \langle u \rangle^\natural} \tilde{\lambda}(x)^{-1} \tilde{\mu}(y)^{-1} kG_{\lambda\mu}. \end{aligned}$$

Setting $\rho = \lambda\mu$, we get

$$\begin{aligned} E_{L,u}^G(x) \cdot E_{L,u}^G(y) &= \frac{1}{|u|^2} \sum_{\lambda, \rho \in \langle u \rangle^\natural} \tilde{\lambda}(x)^{-1} \tilde{\lambda}(y) \tilde{\rho}(y)^{-1} kG_\rho \\ &= \frac{1}{|u|^2} \sum_{\rho \in \langle u \rangle^\natural} \left(\sum_{\lambda \in \langle u \rangle^\natural} \tilde{\lambda}(x)^{-1} \tilde{\lambda}(y) \right) \tilde{\rho}(y)^{-1} kG_\rho. \end{aligned}$$

The inner sum is equal to $|u|$ if $x = y$, and equal to zero otherwise. This shows that the elements $E_{L,u}^G$, for $x \in \langle u \rangle$ are pairwise orthogonal idempotents of $RT^\Delta(G, G)$. Now the fact that their sum is equal to the identity element kG follows from the observation that for any $\lambda \in \langle u \rangle^\natural$, one has

$$\frac{1}{|u|} \sum_{x \in \langle u \rangle} \tilde{\lambda}(x)^{-1} = \begin{cases} 1, & \text{if } \lambda = 1 \\ 0, & \text{otherwise.} \end{cases}$$

□

8.3 Proposition *Let (L, u) be a D^Δ -pair. Then the functor $RT^\Delta(-, L\langle u \rangle)E_{L,u}$ is compact and projective in $\mathcal{F}_{Rpp_k}^\Delta$.*

Proof The proof is similar to the proof of Lemma 6.3 in [6].

□

8.4 Proposition Every functor in $\mathcal{F}_{Rpp_k}^\Delta$ is a quotient of a direct sum of representable functors $RT^\Delta(-, L\langle u \rangle)E_{L,u}$.

Proof The proof is similar to the proof of Proposition 6.4 in [6]. We add some details for convenience.

Since representable functors generate $\mathcal{F}_{Rpp_k}^\Delta$, it suffices to prove the statement for representable functors $RT^\Delta(-, G)$. By induction on $|G|$, we can assume that the essential algebra $\mathcal{E}_R(G) \neq 0$, i.e., $G = L\langle u \rangle$ for some D^Δ -pair (L, u) .

Now by Lemma 8.2, the identity element kG is equal to the sum of idempotents $E_{L,u}^G(x)$ for $x \in \langle u \rangle$. If x does not generate $\langle u \rangle$, then

$$E_{L,u}^G(x) = \frac{1}{|u|} \sum_{\mu \in \langle x \rangle^\natural} \tilde{\mu}(x)^{-1} \left(\sum_{\substack{\lambda \in \langle u \rangle^\natural \\ \text{Res}_{\langle x \rangle}^{\langle u \rangle} \lambda = \mu}} kG_\lambda \right)$$

vanishes in $\mathcal{E}_R(G)$ by the proof of Theorem 4.9 in [8]. If x generates $\langle u \rangle$, then the functors $RT^\Delta(-, G)E_{L,u}^G(x)$ and $RT^\Delta(-, G)E_{L,x}^G$ are equal since $E_{L,u}^G(x) = E_{L,x}^G$. The result follows. $\square$

8.5 Definition Let RD^Δ denote the following category:

- The objects are D^Δ -pairs.
- $\text{Hom}_{RD^\Delta}((L, u), (M, v)) = E_{M,v}RT^\Delta(M\langle v \rangle, L\langle u \rangle)E_{L,u}$.
- The composition is induced from tensor product of bimodules.
- $\text{Id}_{L,u} = E_{L,u}kL\langle u \rangle E_{L,u}$.

8.6 Corollary The category $\mathcal{F}_{Rpp_k}^\Delta$ of diagonal p -permutation functors over R is equivalent to the category of R -linear functors from RD^Δ to $R\text{Mod}$.

Proof This follows from Propositions 8.3, 8.4 and [6, Theorem 5.1]. $\square$

Our goal is to prove that the category $\mathcal{F}_{Rpp_k}^\Delta$ of diagonal p -permutation functors over R is equivalent to the category of R -linear functors on a quotient of the category $R\mathcal{P}^\Delta$ of diagonal D^Δ -pair bisets over R .

We will prove this by constructing an ideal $\mathcal{I}^\Delta$ in $R\mathcal{P}^\Delta$ and by showing that the quotient category

$$\overline{R\mathcal{P}^\Delta} := R\mathcal{P}^\Delta / \mathcal{I}^\Delta$$

is isomorphic to RD^Δ . The remainder of the paper is devoted to the construction of this isomorphism. We begin in the next section by constructing a convenient basis for

$$E_{M,v}RT^\Delta(M\langle v \rangle, L\langle u \rangle)E_{L,u}.$$

9 A basis

Throughout this section we fix two D^Δ -pairs (M, v) and (L, u) . The goal of this section is to give an R -basis for $E_{M,v}RT^\Delta(M\langle v \rangle, L\langle u \rangle)E_{L,u}$.

Recall that, for a finite group G , we denote by $T(G)$ the trivial source ring of G , i.e., the Grothendieck group of p -permutation kG -modules with respect to direct sums and that $RT(G) = R \otimes_{\mathbb{Z}} T(G)$.

9.1 Notation (a) For any finite group G , we denote by $RG^\natural$ the R -submodule of $RT(G)$ spanned by the elements k_λ , $\lambda \in G^\natural$. Moreover we set

$$\chi_g := \chi_g^G := \frac{1}{|G^\natural|} \sum_{\lambda \in G^\natural} \tilde{\lambda}(g^{-1})k_\lambda \in RG^\natural \subseteq RT(G).$$

(b) Let $Q = \Delta(Y, \alpha, X) \leq M \times L$ be a diagonal subgroup which is invariant by (v, u) . We define

$$\Psi_{L,u}^{M,v}(Q) := \frac{|v|}{\gcd(|v|, |u|)} \text{Ind}_{Q \cdot \langle (v,u) \rangle}^{M\langle v \rangle \times L\langle u \rangle} (\chi_{(v,u)}^{Q \cdot \langle (v,u) \rangle}) \in RT^\Delta(M\langle v \rangle, L\langle u \rangle).$$

9.2 Remark Let G be a normal Sylow p -subgroup P with abelian quotient G/P .

(a) Clearly $|G^\natural| = [G : P] = |G|_{p'}$ and one has an R -module isomorphism

$$RG^\natural \longrightarrow F_P(G, R), \quad k_\lambda \mapsto \tilde{\lambda},$$

between the R -scalar extension of the Grothendieck group $T(G)$ of p -permutation kG -modules and the R -module of functions from G to R which are constant on cosets $gP \in G/P$. It is easy to verify that under the above isomorphism the element χ_g^G corresponds to the characteristic function on the coset gP .

(b) If S is a complement of P in G , one has group isomorphisms $(G/P)^\natural \rightarrow G^\natural$ and $G^\natural \rightarrow S^\natural$ given by inflation and restriction. They induce R -module isomorphisms

$$\text{Inf}_{G/P}^G : R(G/P)^\natural \rightarrow RG^\natural \quad \text{and} \quad \text{Res}_S^G : RG^\natural \rightarrow RS^\natural.$$

(c) Let H be a subgroup of G . Then $Q := H \cap P$ is a normal Sylow p -subgroup of H and $H/Q \cong HP/P$ is abelian. Moreover, the group homomorphism $G^\natural \rightarrow H^\natural$, $\lambda \mapsto \lambda|_H$, is surjective and every element $\mu \in H^\natural$ has precisely $[G : H]_{p'} = |G^\natural|/|H^\natural|$ preimages. This implies that, for all $g \in G$, one has

$$\text{Res}_H^G(\chi_g^G) = \begin{cases} \chi_h^H, & \text{if } g \in PH, \\ 0, & \text{otherwise} \end{cases}$$

where $h \in H$ is such that $hP = gP$, and that

$$\text{Ind}_H^G(\chi_h^H) = \text{Ind}_H^G(\text{Res}_H^G(\chi_h^G)) = \chi_h^G \cdot \text{Ind}_H^G(k),$$

for all $h \in H$.

(d) It is a straightforward computation that, for any $\theta \in G^{\natural}$ and $g \in G$, one has

$$k_{\theta} \cdot \chi_g = \tilde{\theta}(g) \cdot \chi_g.$$

(e) Note that Parts (a)–(c) apply to the groups $L\langle u \rangle$, $M\langle v \rangle$, and $M\langle v \rangle \times L\langle u \rangle$. In particular, if $Q = \Delta(Y, \alpha, X) \leq M \times L$ is a diagonal subgroup which is invariant by (v, u) then

$$\begin{aligned} \Psi_{L,u}^{M,v}(Q) &= \frac{|v|}{\gcd(|v|, |u|)} \text{Ind}_{Q \cdot \langle (v,u) \rangle}^{M\langle v \rangle \times L\langle u \rangle} (\chi_{(v,u)}^{Q \cdot \langle (v,u) \rangle}) \\ &= \frac{|v|}{\gcd(|v|, |u|)} \chi_{(v,u)}^{M\langle v \rangle \times L\langle u \rangle} \cdot \text{Ind}_{Q \cdot \langle (v,u) \rangle}^{M\langle v \rangle \times L\langle u \rangle}(k). \end{aligned}$$

9.3 Lemma *Let W be a $(kM\langle v \rangle, kL\langle u \rangle)$ -bimodule, let $\mu \in \langle v \rangle^{\natural}$ and let $\lambda \in \langle u \rangle^{\natural}$. Let also $W_{\mu,\lambda}$ be the $(kM\langle v \rangle, kL\langle u \rangle)$ -bimodule equal to W as a k -vector space and with the action*

$$b \cdot w \cdot a = \mu(b)\lambda(a)^{-1}bwa,$$

for $b \in M\langle v \rangle, a \in L\langle u \rangle$ and $w \in W$. Then we have an isomorphism

$$(kM\langle v \rangle)_{\mu} \otimes_{kM\langle v \rangle} W \otimes_{kL\langle u \rangle} (kL\langle u \rangle)_{\lambda} \cong W_{\mu,\lambda}$$

of $(kM\langle v \rangle, kL\langle u \rangle)$ -bimodules.

Proof The map

$$\begin{aligned} \varphi : (kM\langle v \rangle)_{\mu} \otimes_{kM\langle v \rangle} W \otimes_{kL\langle u \rangle} (kL\langle u \rangle)_{\lambda} &\rightarrow W_{\mu,\lambda} \\ 1 \otimes w \otimes 1 &\mapsto w \end{aligned}$$

is clearly an isomorphism of k -vector spaces. For $b \in M\langle v \rangle, a \in L\langle u \rangle$ and $w \in W$, one has

$$\begin{aligned} \varphi(b \cdot (1 \otimes w \otimes 1) \cdot a) &= \varphi((b \cdot 1) \otimes w \otimes (1 \cdot a)) = \mu(b)\lambda(a)^{-1}\varphi((1 \cdot b \otimes w \otimes a \cdot 1)) \\ &= \mu(b)\lambda(a)^{-1}\varphi((1 \otimes bwa \otimes 1)) \\ &= \mu(b)\lambda(a)^{-1}bwa. \end{aligned}$$

This proves that the map φ is an isomorphism of $(kM\langle v \rangle, kL\langle u \rangle)$ -bimodules. $\square$

9.4 Lemma *Let $H \leq M\langle v \rangle \times L\langle u \rangle$ and let $\theta \in H^{\natural}$. Moreover, let $\mu \in M\langle v \rangle^{\natural}$ and $\lambda \in L\langle u \rangle^{\natural}$. Then the $(kM\langle v \rangle, kL\langle u \rangle)$ -bimodule*

$$(kM\langle v \rangle)_{\mu} \otimes_{kM\langle v \rangle} \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle}(k_{\theta}) \otimes_{kL\langle u \rangle} (kL\langle u \rangle)_{\lambda}$$

when viewed as $k(M\langle v \rangle \times L\langle u \rangle)$ -module is isomorphic to

$$\text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle}(k_{\theta(\mu \times \lambda)|_H}).$$

Proof By Lemma 9.3, the bimodule

$$(kM\langle v \rangle)_\mu \otimes_{kM\langle v \rangle} \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (k_\theta) \otimes_{kL\langle u \rangle} (kL\langle u \rangle)_\lambda$$

when viewed as $k(M\langle v \rangle \times L\langle u \rangle)$ -module is isomorphic to

$$k_{\mu \times \lambda} \otimes_k \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (k_\theta) \cong \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (k_{\theta(\mu \times \lambda)|_H}).$$

□

9.5 Lemma Let $H \leq M\langle v \rangle \times L\langle u \rangle$ and let $\theta \in H^\natural$. Then

$$E_{M,v} \left(\text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (k_\theta) \right) E_{L,u} = \begin{cases} \tilde{\theta}(h) \cdot \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (\chi_h) & \text{if } (v, u) \in (M \times L)H \\ 0 & \text{if } (v, u) \notin (M \times L)H, \end{cases}$$

where $h \in H$ is such that $h \in (M \times L)(v, u)$.

Proof By Lemma 9.4, we have

$$\begin{aligned} & E_{M,v} \left(\text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (k_\theta) \right) E_{L,u} \\ &= \frac{1}{|v||u|} \sum_{\substack{\mu \in M\langle v \rangle^\natural \\ \lambda \in M\langle v \rangle^\natural}} \tilde{\mu}(v)^{-1} \tilde{\lambda}(u)^{-1} \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (k_{\theta(\mu \times \lambda)|_U}) \\ &= \frac{1}{|v||u|} \sum_{\substack{\mu \in M\langle v \rangle^\natural \\ \lambda \in M\langle v \rangle^\natural}} k_{\mu \times \lambda} \cdot \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (k_\theta) \\ &= \chi_{(v,u)}^{M\langle v \rangle \times L\langle u \rangle} \cdot \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (k_\theta) \\ &= \text{Ind}_H^{M\langle v \rangle \times L\langle u \rangle} (\text{Res}_H^{M\langle v \rangle \times L\langle u \rangle} (\chi_{(v,u)}^{M\langle v \rangle \times L\langle u \rangle}) \cdot k_\theta). \end{aligned}$$

By Remark 9.2(c), $\text{Res}_H^{M\langle v \rangle \times L\langle u \rangle} (\chi_{(v,u)}^{M\langle v \rangle \times L\langle u \rangle}) = 0$ if $(v, u) \notin (M \times L)H$ and equal to χ_w^H for any $w \in H$ with $w \in (M \times L)H$. Further, by Remark 9.2(d), we have $\chi_w^H \cdot k_\theta = \tilde{\theta}(w) \cdot \chi_w^H$. This completes the proof. □

9.6 Theorem The elements

$$\Psi_{L,u}^{M,v}(Q) \in E_{M,v} RT^\Delta(M\langle v \rangle, L\langle u \rangle) E_{L,u},$$

where the subgroup $Q = \Delta(Y, \alpha, X)$ runs through a set of representatives of the $M^v\langle v \rangle \times L^u\langle u \rangle$ -conjugacy classes of (v, u) -invariant diagonal subgroups of $M\langle v \rangle \times L\langle u \rangle$, form an R -basis for $E_{M,v} RT^\Delta(M\langle v \rangle, L\langle u \rangle) E_{L,u}$.

Proof Set $G := M\langle v \rangle \times L\langle u \rangle$. Let $Q \leq M \times L$ be a diagonal subgroup, let U be a Hall p' -subgroup of $N_G(Q)$, and set $H := QU$. By [2, Theorem 4.1], the isomorphism classes of indecomposable p -permutation kG -modules with vertex Q are represented by the modules

$$\text{Ind}_H^G(k_\theta),$$

where θ runs through $H^\natural$. By Lemma 9.5, the product

$$E_{M,v}(\text{Ind}_H^G(k_\theta))E_{L,u}$$

equals 0 unless $(v, u) \in (M \times L)H = (M \times L)U$. After conjugating Q with an element in $(M \times L)U$ if necessary, we may assume that $(v, u) \in U$. Thus, if none of the G -conjugates of Q is (v, u) -invariant, then $E_{M,v}WE_{L,u} = 0$ for all indecomposable p -permutation kG -modules W with vertex Q .

Assume now that Q is (v, u) -invariant and H is chosen such that $(v, u) \in H$. Then, by Lemma 9.5 and Remark 9.2(c), we have

$$\begin{aligned} E_{M,v}(\text{Ind}_H^G(k_\theta))E_{L,u} &= \tilde{\theta}(v, u) \cdot \text{Ind}_H^G(\chi_{v,u}^H) \\ &= \tilde{\theta}(v, u) \cdot \text{Ind}_{Q\langle(v,u)\rangle}^G(\chi_{v,u}^{Q\langle(v,u)\rangle}). \end{aligned}$$

Note that this element depends up to the scalar $\tilde{\theta}(v, u)$ only on Q and not on θ .

Let $\mathcal{S}$ be the set of diagonal subgroups of $M \times L$ and let $\mathcal{S}'$ denote the set of diagonal subgroups of $M \times L$ which have a G -conjugate that is (v, u) -invariant. So far, we have proved that the elements

$$\text{Ind}_{Q\langle(v,u)\rangle}^G(\chi_{v,u}^{Q\langle(v,u)\rangle}),$$

where Q runs through a set of representatives of the G -conjugacy classes of $\mathcal{S}'$, generate the R -module $E_{M,v}RT^\Delta(M\langle v \rangle, L\langle u \rangle)E_{L,u}$. Let $\mathcal{S}^{(v,u)}$ denote the subset of $\mathcal{S}$ consisting of (v, u) -invariant subgroups in $\mathcal{S}$ and let $[\mathcal{S}^{(v,u)}]$ be a set of representative of the $M^v \times L^u$ -conjugacy classes of $\mathcal{S}^{(v,u)}$. Then, by Glauberman's Lemma, $[\mathcal{S}^{(v,u)}]$ is also a set of representatives of the G -conjugacy classes of $\mathcal{S}'$. Thus, the elements $\Psi_{(L,u)}^{(M,v)}(Q)$, with $Q \in [\mathcal{S}^{(v,u)}]$ generate the R -module $E_{M,v}RT^\Delta(M\langle v \rangle, L\langle u \rangle)E_{L,u}$.

Now, suppose that

$$\sum_{Q \in [\mathcal{S}^{(v,u)}]} a_Q \Psi_{L,u}^{M,v}(Q) = 0$$

with $a_Q \in R$, and suppose that not all the coefficients a_Q are zero. Choose $Q_0 \in [\mathcal{S}^{(v,u)}]$ such that $a_{Q_0} \neq 0$ and $|Q_0|$ is maximal among all $Q \in [\mathcal{S}^{(v,u)}]$ with $a_Q \neq 0$.

We apply the R -linear extension of the Brauer quotient at Q_0 to the above equation. For $Q \in [\mathcal{S}^{(v,u)}]$, the element $\Psi_{L,u}^{M,v}(Q)$ is a non-zero scalar multiple of

$$\text{Ind}_{Q\langle(v,u)\rangle}^G(\chi_{v,u}),$$

which is relatively Q -projective. Hence its Brauer quotient at Q_0 can be non-zero only if Q_0 is G -conjugate to a subgroup of Q . By the maximality of $|Q_0|$, this can happen only if Q_0 and Q are G -conjugate. Since $[\mathcal{S}^{v,u}]$ has been chosen as a set of representatives of the G -conjugacy classes of $\mathcal{S}'$, this forces $Q = Q_0$. Thus

$$\mathrm{Br}_{Q_0}(\Psi_{L,u}^{M,v}(Q)) = 0 \quad \text{for all } Q \in [\mathcal{S}^{v,u}], Q \neq Q_0.$$

On the other hand, again by [2, Theorem 4.1], one has

$$\mathrm{Br}_{Q_0}(\Psi_{L,u}^{M,v}(Q_0)) = \kappa \cdot \mathrm{Ind}_{Q_0 \langle (v,u) \rangle}^{N_G(Q_0)}(\chi_{v,u}) \neq 0,$$

with a non-zero constant $\kappa \in R$. This implies that $a_{Q_0} = 0$, a contradiction. Therefore all coefficients a_Q are zero, and the elements

$$\Psi_{L,u}^{M,v}(Q), \quad Q \in [\mathcal{S}^{v,u}],$$

are R -linearly independent. □

10 Composition formula

The main goal of this section is a formula (see Theorem 10.3) that expresses the composition of two basis elements from the last section again in terms of this basis.

10.1 Lemma *Let $a, b, c \in \mathbb{N}$. One has*

$$\mathrm{gcd}\left(\frac{b}{\mathrm{gcd}(a,b)}, \frac{b}{\mathrm{gcd}(c,b)}\right) = \frac{b}{\mathrm{lcm}(\mathrm{gcd}(a,b), \mathrm{gcd}(c,b))}.$$

Proof This follows from the fact that the map $i \mapsto b/i$ is an isomorphism from the lattice of divisors of b (ordered by divisibility) to its opposite lattice. □

For the definition of the notation used in the following Lemma we refer the reader to [3, Notations 2.3.19, 2.3.21]

10.2 Lemma *Let $(N, w), (M, v)$ and (L, u) be D^Δ -pairs, let $\Delta(T, \beta, Z) \leq N \times M$ be a (w, v) -invariant subgroup and $\Delta(Y, \alpha, X) \leq M \times L$ be a (v, u) -invariant subgroup. For any $m \in M$, set*

$$K_m := k_2(\Delta(T, \beta, Z) \cdot \langle (w, v) \rangle) \cap {}^m k_1(\Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle) \leq M \langle v \rangle$$

and

$$\Delta(T, *, X)_m := (\Delta(T, \beta, Z) \cdot \langle (w, v) \rangle) * {}^{(m,1)}(\Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle) \leq N \langle w \rangle \times L \langle u \rangle.$$

(a) One has

$$K_m = \{v^j \in C_{\langle v \rangle}(m) \mid \exists i \in \mathbb{Z}: v^i = v^j, w^j = 1, u^i = 1\},$$

and if $m \in M^v$, then

$$|K_m| = \frac{|v| \cdot \gcd(|w|, |v|, |u|)}{\gcd(|w|, |v|) \cdot \gcd(|v|, |u|)}.$$

(b) The group $\Delta_m := \Delta(\beta(Z \cap {}^m Y), \beta i_m \alpha, \alpha^{-1}(Y \cap Z^m))$ is a Sylow p -subgroup of $\Delta(T, *, X)_m$.

(c) If $m \in M^v$, then

$$\Delta(T, *, X)_m = \Delta_m \cdot (\langle (w, v) \rangle * \langle (v, u) \rangle).$$

(d) One has

$$|\langle (w, v) \rangle * \langle (v, u) \rangle| = \frac{|w| \cdot |u|}{\gcd(|w|, |v|, |u|)}.$$

Proof (a) We have

$$\begin{aligned} K_m &= \{b \in M\langle v \rangle \mid (1, b) \in \Delta(T, \beta, Z) \cdot \langle (w, v) \rangle, (b^m, 1) \in \Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle\} \\ &= \{b \in M\langle v \rangle \mid \exists z \in Z, x \in X, i, j \in \mathbb{Z}, (1, b) \in (\beta(z), z)(w^j, v^j), (b^m, 1) = (\alpha(x), x)(v^i, u^i)\} \\ &= \{v^j \mid w^j = 1, (v^j)^m = v^i, u^i = 1\} \\ &= \{v^j \mid w^j = 1, [m, v^j]v^j = v^i, u^i = 1\} \\ &= \{v^j \in C_{\langle v \rangle}(m) \mid \exists i \in \mathbb{Z}: v^i = v^j, w^j = 1, u^i = 1\}, \end{aligned}$$

which proves the first part. Now assume that $m \in M^v$. Then

$$\begin{aligned} K_m &= \{v^j \mid \exists i \in \mathbb{Z}: v^i = v^j, u^i = 1, w^i = 1\} \\ &= \{v^j \mid w^j = 1\} \cap \{v^i \mid u^i = 1\} \\ &= \langle v^{|w|} \rangle \cap \langle v^{|u|} \rangle \end{aligned}$$

which together with Lemma 10.1 implies that

$$\begin{aligned} |K_m| &= \gcd\left(\frac{|v|}{\gcd(|w|, |v|)}, \frac{|v|}{\gcd(|v|, |u|)}\right) = \frac{|v|}{\text{lcm}(\gcd(|w|, |v|), \gcd(|v|, |u|))} \\ &= \frac{|v| \cdot \gcd(|w|, |v|, |u|)}{\gcd(|w|, |v|) \cdot \gcd(|v|, |u|)}. \end{aligned}$$

(b) Let $(c, a) \in (N \times L) \cap \Delta(T, *, X)_m$, the Sylow p -subgroup of $\Delta(T, *, X)_m$. Then there exists $b \in M\langle v \rangle$ such that $(c, b) \in (\Delta(T, \beta, Z) \cdot \langle (w, v) \rangle)$ and $(b^m, a) \in (\Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle)$. Thus,

$$(c, b) = (\beta(z), z)(w^j, v^j) \quad \text{and} \quad (b^m, a) = (\alpha(x), x)(v^i, u^i)$$

for some $z \in Z$, $x \in X$ and integers i and j . Since $c \in N$, the equation $c = \beta(z)w^j$ implies that $w^j = 1$ and $c = \beta(z)$. Similarly, one has $a = x$ and $u^i = 1$. Moreover,

$$zv^j = b = {}^m(\alpha(x)v^i) = {}^m\alpha(x)[m, v^i]v^i$$

which implies that $v^j = v^i$. Furthermore, $w^j = 1$ implies that $v^j \in k_1(\langle (w, v) \rangle)$ and in particular, v^j centralizes Z . So, z is the p -part of b and v^j is the p' -part of b . Similarly, $u^i = 1$ implies that v^i centralizes Y and so ${}^m\alpha(x)$ is the p -part of b and ${}^mv^i$ is the p' -part of b . These imply that $z = {}^m\alpha(x)$ and $v^j = {}^mv^i = v^i$. This shows that $(c, a) \in \Delta_m$. The converse inclusion holds trivially.

(c) Let $(c, a) \in \Delta(T, *, X)_m$. Then there exists $b \in M\langle v \rangle$ such that $(c, b) \in \Delta(T, \beta, Z)\langle (w, v) \rangle$ and $(b^m, a) \in \Delta(Y, \alpha, X)\langle (v, u) \rangle$. Thus, we have

$$(c, b) = (\beta(z)w^j, zv^j) \quad \text{and} \quad (b^m, a) = (\alpha(x)v^i, xu^i)$$

for some $z \in Z$, $x \in X$, $i, j \in \mathbb{Z}$. Since m is $\langle v \rangle$ -fixed, we obtain that

$$b = zv^j = {}^m\alpha(x){}^mv^i = {}^m\alpha(x)v^i$$

which shows that $v^j = v^i$ and $z = {}^m\alpha(x)$. This proves the claim.

(d) Write $D = \langle (w, v) \rangle * \langle (v, u) \rangle$. Note that

$$\begin{aligned} k_1(D) &= \{w^a \in \langle w \rangle \mid \exists b \in \mathbb{Z}, v^a = v^b, u^b = 1\} \\ &= \{w^a \in \langle w \rangle \mid \exists k, l \in \mathbb{Z}, a = k|u| + l|v|\} \\ &= \langle w^{(|v|, |u|)} \rangle, \end{aligned}$$

and hence

$$|k_1(D)| = \frac{|w|}{(|w|, |v|, |u|)}.$$

We also have $p_2(D) = \langle u \rangle$. Therefore,

$$|D| = |k_1(D)| \cdot |p_2(D)| = \frac{|w| \cdot |u|}{(|w|, |v|, |u|)},$$

as desired. □

10.3 Theorem (Composition Formula) Let $(N, w), (M, v)$ and (L, u) be D^Δ -pairs, let $\Delta(T, \beta, Z) \leq N \times M$ be a (w, v) -invariant subgroup and $\Delta(Y, \alpha, X) \leq M \times L$ be a (v, u) -invariant subgroup. Then

$$\Psi_{M,v}^{N,w}(\Delta(T, \beta, Z)) \circ \Psi_{L,u}^{M,v}(\Delta(Y, \alpha, X)) = \sum_{m \in Z^v \backslash M^v / Y^v} \Psi_{L,u}^{N,w}(\Delta_m)$$

where $\Delta_m = \Delta(\beta(Z \cap {}^m Y), \beta i_m \alpha, \alpha^{-1}(Y \cap Z^m))$.

Proof By definition, the composition

$$\Psi_{M,v}^{N,w}(\Delta(T, \beta, Z)) \circ \Psi_{L,u}^{M,v}(\Delta(Y, \alpha, X))$$

is equal to

$$\lambda \cdot \text{Ind}_{\Delta(T, \beta, Z) \cdot \langle (w, v) \rangle}^{N \langle w \rangle \times M \langle v \rangle} (\chi_{(w, v)}) \otimes_{kM \langle v \rangle} \text{Ind}_{\Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle}^{M \langle v \rangle \times L \langle u \rangle} (\chi_{(v, u)}),$$

where $\lambda = \frac{|w||v|}{(|w|, |v|)(|v|, |u|)}$. By Theorem 1.1 of [4], we have

$$\begin{aligned} & \text{Ind}_{\Delta(T, \beta, Z) \cdot \langle (w, v) \rangle}^{N \langle w \rangle \times M \langle v \rangle} (\chi_{(w, v)}) \otimes_{kM \langle v \rangle} \text{Ind}_{\Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle}^{M \langle v \rangle \times L \langle u \rangle} (\chi_{(v, u)}) \\ &= \sum_{m \in Z \langle v \rangle \backslash M \langle v \rangle / Y \langle v \rangle} \text{Ind}_{\Delta(T, *, X)_m}^{N \langle w \rangle \times L \langle u \rangle} (\chi_{(w, v)} \otimes_{K_m} {}^{(m, 1)} \chi_{(v, u)}), \end{aligned} \quad (2)$$

where we can choose $m \in M$ and where

$$\Delta(T, *, X)_m = (\Delta(T, \beta, Z) \cdot \langle (w, v) \rangle) * {}^{(m, 1)}(\Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle)$$

and

$$K_m = k_2(\Delta(T, \beta, Z) \cdot \langle (w, v) \rangle) \cap {}^m k_1(\Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle).$$

First note that the subgroups Y and Z are v -invariant and that $\langle v \rangle$ acts on the set $Z \backslash M / Y$ by $v \cdot (ZmY) := v(ZmY) = Zv(m)Y$, for $m \in M$. Moreover, if $Zm'Y = Zv(m)Y$ then $Z \langle v \rangle m'Y \langle v \rangle = Z \langle v \rangle mY \langle v \rangle$ in $Z \langle v \rangle \backslash M \langle v \rangle / Y \langle v \rangle$. Therefore, the function

$$\langle v \rangle \backslash (Z \backslash M / Y) \longrightarrow Z \langle v \rangle \backslash M \langle v \rangle / Y \langle v \rangle, \quad (3)$$

induced by $ZmY \mapsto Z \langle v \rangle mY \langle v \rangle$ is well-defined. It is obviously surjective. Moreover, it is injective. In fact, suppose that m, m' are in M with $m' \in Z \langle v \rangle mY \langle v \rangle$. Then $m' = zv^j m y v^i = zv^j(m) v^j(y) v^{j+i}$ for some $i, j \in \mathbb{Z}$, $z \in Z$ and $y \in Y$. But this implies $v^{j+i} = 1$ and $m' \in Z v^j(m) Y$.

We claim that, for any $m \in M$, the element

$$\chi_{(w, v)}^{\Delta(T, \beta, Z) \cdot \langle (w, v) \rangle} \otimes_{K_m} {}^{(m, 1)} \chi_{(v, u)}^{\Delta(Y, \alpha, X) \cdot \langle (v, u) \rangle} \in RT(\Delta(T, *, X)_m) \quad (4)$$

from Equation (2) is equal to 0 if ZmY is not v -stable and equal to

$$\frac{1}{|K_m|} \chi_{(w,u)}^{\Delta(T,*,X)_m}$$

if $m \in M^v$. This will be proved in the subsequent Lemma 10.4.

Thus, we may assume that ZmY is v -stable. By Glauberman's Lemma applied to the action of $(Z \times Y)\langle v \rangle$ on M given by $(z, y)v^i \cdot m := zv^i(m)y^{-1}$, we obtain a bijection $Z^v \backslash M^v / Y^v \rightarrow (Z \backslash M / Y)^v$ onto the fixed points of v on $Z \backslash M / Y$. Combining this with the bijection (3), we can replace the sum over $m \in Z\langle v \rangle \backslash M\langle v \rangle / Y\langle v \rangle$ in the first paragraph by a sum over $m \in Z^v \backslash M^v / Y^v$. Moreover, if $m \in M^v$, then by Lemma 10.2(c) we have

$$\Delta(T, *, X)_m = \Delta_m \cdot (\langle (w, v) \rangle * \langle (v, u) \rangle) .$$

Altogether, we now have

$$\Psi_{M,v}^{N,w}(\Delta(T, \beta, Z)) \circ \Psi_{L,u}^{M,v}(\Delta(Y, \alpha, X)) = \sum_{m \in Z^v \backslash M^v / Y^v} \frac{\lambda}{|K_m|} \text{Ind}_{\Delta(T,*,X)_m}^{N\langle w \rangle \times L\langle u \rangle}(\chi_{(w,u)})$$

which is equal to

$$\sum_{m \in Z^v \backslash M^v / Y^v} \frac{\lambda \cdot |\langle (w, u) \rangle|}{|K_m| \cdot |\langle (w, v) \rangle * \langle (v, u) \rangle|} \text{Ind}_{\Delta_m \cdot \langle (w, u) \rangle}^{N\langle w \rangle \times L\langle u \rangle}(\chi_{(w,u)}) .$$

By Lemma 10.2, one has

$$|\langle (w, v) \rangle * \langle (v, u) \rangle| = \frac{|w| \cdot |u|}{\gcd(|w|, |v|, |u|)} \quad \text{and} \quad |K_m| = \frac{|v| \cdot \gcd(|w|, |v|, |u|)}{\gcd(|w|, |v|) \cdot \gcd(|v|, |u|)} ,$$

and so

$$\begin{aligned} |K_m| |\langle (w, v) \rangle * \langle (v, u) \rangle| : |\langle (w, u) \rangle| &= \frac{|v| \cdot \gcd(|w|, |v|, |u|)}{\gcd(|w|, |v|) \cdot \gcd(|v|, |u|)} \cdot \frac{\gcd(|w|, |u|)}{\gcd(|w|, |v|, |u|)} \\ &= \frac{|v| \cdot \gcd(|w|, |u|)}{\gcd(|w|, |v|) \cdot \gcd(|v|, |u|)} . \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\lambda \cdot |\langle (w, u) \rangle|}{|K_m| \cdot |\langle (w, v) \rangle * \langle (v, u) \rangle|} &= \frac{|w| \cdot |v|}{\gcd(|w|, |v|) \cdot \gcd(|v|, |u|)} \cdot \frac{\gcd(|w|, |v|) \cdot \gcd(|v|, |u|)}{|v| \cdot \gcd(|w|, |u|)} \\ &= \frac{|w|}{\gcd(|w|, |u|)} . \end{aligned}$$

Therefore,

$$\begin{aligned}\Psi_{M,v}^{N,w}(\Delta(T, \beta, Z)) \circ \Psi_{L,u}^{M,v}(\Delta(Y, \alpha, X)) &= \sum_{m \in Z^v \setminus M^v / Y^v} \frac{|w|}{\gcd(|w|, |u|)} \text{Ind}_{\Delta_m \cdot \langle (w,u) \rangle}^{N\langle w \rangle \times L\langle u \rangle}(\chi_{(w,u)}) \\ &= \sum_{m \in Z^v \setminus M^v / Y^v} \Psi_{L,u}^{N,w}(\Delta_m)\end{aligned}$$

as was to be shown. $\square$

10.4 Lemma *Assume the hypotheses and notation from Theorem 10.3 and let $m \in M$. Then the element*

$$\chi_{w,v} \otimes_{K_m} {}^{(m,1)}\chi_{v,u} \in RT(\Delta(T, *, X)_m)$$

is equal to 0 if ZmY is not v -invariant and it is equal to $\frac{1}{|K_m|}\chi_{w,u}$ if $m \in M^v$.

Proof We abbreviate

$$H_\alpha := \Delta(Y, \alpha, X) \langle (v, u) \rangle \quad \text{and} \quad H_\beta := \Delta(T, \beta, Z) \langle (w, v) \rangle.$$

Then, since $\chi_{(w,v)}^{H_\beta} = \text{Res}_{H_\beta}^{N\langle w \rangle \times M\langle v \rangle}(\chi_{(w,v)}^{N\langle w \rangle \times M\langle v \rangle})$ and $\chi_{(v,u)}^{H_\alpha} = \text{Res}_{H_\alpha}^{M\langle v \rangle \times L\langle u \rangle}(\chi_{(v,u)}^{M\langle v \rangle \times L\langle u \rangle})$, we have

$$\begin{aligned}& |w||v|^2|u| \cdot \left(\chi_{(w,v)}^{H_\beta} \otimes_{K_m} {}^{(m,1)}\chi_{(v,u)}^{H_\alpha} \right) \\ &= \left(\sum_{\substack{\nu \in N\langle w \rangle^\natural \\ \mu' \in M\langle v \rangle^\natural}} \tilde{\nu}(w)^{-1} \tilde{\mu}'(v)^{-1} k_{(\nu \times \mu')|_{H_\beta}} \right) \otimes_{K_m} {}^{(m,1)} \left(\sum_{\substack{\mu \in M\langle v \rangle^\natural \\ \lambda \in L\langle u \rangle^\natural}} \tilde{\mu}(v)^{-1} \tilde{\lambda}(u)^{-1} k_{(\mu \times \lambda)|_{H_\alpha}} \right) \\ &= \sum_{\nu, \mu', \mu, \lambda} \tilde{\nu}(w)^{-1} \tilde{\mu}'(v)^{-1} \tilde{\mu}(v)^{-1} \tilde{\lambda}(u)^{-1} \left(k_{(\nu \times \mu')|_{H_\beta}} \otimes_{K_m} k_{({}^m\mu \times \lambda)|_{({}^{(m,1)}H_\alpha)}} \right).\end{aligned} \tag{5}$$

Moreover, by Theorem 1.1 of [4],

$$k_{(\nu \times \mu')|_{H_\beta}} \otimes_{K_m} k_{({}^m\mu \times \lambda)|_{({}^{(m,1)}H_\alpha)}} = \begin{cases} 0, & \text{if } (\mu' \cdot {}^m\mu)|_{K_m} \neq 1, \\ k_{\rho_{\nu, \mu', \mu, \lambda}}, & \text{if } (\mu' \cdot {}^m\mu)|_{K_m} = 1, \end{cases}$$

where $\rho_{\nu, \mu', \mu, \lambda} \in (H_\beta * {}^{(m,1)}H_\alpha)^\natural = \Delta(T, *, X)_m^\natural$ is given by

$$\rho_{\nu, \mu', \mu, \lambda}(c, a) = \nu(c) \mu'(b) {}^m\mu(b) \lambda(a),$$

where $b \in M\langle v \rangle$ is such that

$$(c, b) \in H_\beta \quad \text{and} \quad (b^m, a) \in H_\alpha.$$

Thus, the function

$$f: \Delta(T, *, X)_m \rightarrow R$$

from Remark 9.2(a) associated to the element in (5) is given for $(c, a) \in \Delta(T, *, X)_m$ by

$$\begin{aligned} f(c, a) &= \sum_{\substack{\nu, \mu', \mu, \lambda \\ (\mu' {}^m \mu)|_{K_m} = 1}} \tilde{\nu}(w)^{-1} \tilde{\mu}'(v)^{-1} \tilde{\mu}(v)^{-1} \tilde{\lambda}(u)^{-1} \tilde{\nu}(c) \tilde{\mu}'(b) \tilde{\mu}(m^{-1}bm) \tilde{\lambda}(a) \\ &= \left(\sum_{\nu \in N\langle w \rangle^\natural} \tilde{\nu}(w^{-1}c) \right) \left(\sum_{\lambda \in L\langle u \rangle^\natural} \tilde{\lambda}(u^{-1}a) \right) \left(\sum_{\substack{\mu, \mu' \in M\langle v \rangle^\natural \\ (\mu' {}^m \mu)|_{K_m} = 1}} \tilde{\mu}'(v^{-1}b) \tilde{\mu}(v^{-1}m^{-1}bm) \right) \quad (6) \end{aligned}$$

The first sum is non-zero if and only if $w^{-1}c \in N$, in which case it is equal to $|w|$. The second sum is non-zero if and only if $u^{-1}a \in L$, in which case it is equal to $|u|$. Since

$$\tilde{\mu}(v^{-1}m^{-1}bm) = \tilde{\mu}(v^{-1}b) \quad \text{and} \quad ({}^m \mu)|_{K_m} = \mu|_{K_m},$$

putting $\eta = \mu' \mu$, the third sum becomes

$$|v| \sum_{\substack{\eta \in M\langle v \rangle^\natural \\ \eta|_{K_m} = 1}} \tilde{\eta}(v^{-1}b).$$

Hence, the third sum is non-zero if and only if $v^{-1}b \in MK_m$ which holds if and only if there exists $s \in K_m$ such that $bs \in Mv$. Thus, after replacing b with bs , if the element in (5) is non-zero, there exists $(c, a) \in \Delta(T, *, X)_m$ and $b \in Mv$ such that

$$c \in Nw, \quad a \in Lu, \quad (c, b) \in H_\beta, \quad \text{and} \quad (b^m, a) \in H_\alpha.$$

We claim that this implies that ZmY is v -invariant.

Since $(c, b) \in H_\beta$, $c \in Nw$, and $b \in Mv$, there exists $z \in Z$ such that

$$(c, b) = (\beta(z)w, zv).$$

Similarly, since $(b^m, a) \in H_\alpha$, $b^m \in Mv$ and $a \in Lu$, there exists $x \in X$ such that

$$(b^m, a) = (\alpha(x)v, xu).$$

Further, this implies

$$zv = b = m\alpha(x)vm^{-1} = m\alpha(x)v(m^{-1})v.$$

Thus, $v(m) = z^{-1}m\alpha(x) \in ZmY$, as claimed.

We have therefore shown that if ZmY is not v -invariant then the element in (5) is equal to 0, which is the first statement of the Lemma. Now suppose that $v(m) = m$. By Lemma 10.2(c), it suffices to show that the function $f: \Delta(T, *, X)_m \rightarrow R$ from above

maps elements in $\Delta_m(w, u)$ to $|w||v|^2|u|/|K_m|$ and vanishes on all other elements. We have already seen that f vanish on (c, a) unless $(c, a) \in (N \times L)(w, u)$, which implies $(c, a) \in \Delta(T, *, X)_m \cap (N \times L)(w, u) = \Delta_m(w, u)$ by Lemma 10.2(b) and (c). Thus, it suffices to show that $f(w, u) = |w||v|^2|u|/|K_m|$. But for $(c, a) = (w, u)$ we can choose $b = v$ and Equation (6) gives the desired result, since $v^{-1}m^{-1}vm = v^{-1}(m^{-1})m \in M$ and since the last sum in (6) has $|v|^2/|K_m|$ summands. $\square$

10.5 Notation Let (L, u) be a D^Δ -pair, let $X \leq L$ be v -invariant, and let $u' \in \text{Aut}(L)$ be the restriction of u . then (X, u') is a D^Δ -subpair of (L, u) . We set

$$\text{Res}_{X, u'}^{L, u} := \Psi_{L, u}^{X, u'}(\Delta(X)) \quad \text{and} \quad \text{Ind}_{X, u'}^{L, u} := \Psi_{X, u'}^{L, u}(\Delta(X)),$$

where $\Delta(X) := \Delta(X, \text{id}_X, X)$. Moreover, if (L, u) and (M, v) are D^Δ -pairs and $\alpha: L \rightarrow M$ is an isomorphism of D^Δ -pairs then we define

$$\text{Iso}(\alpha) := \Psi_{L, u}^{M, v}(\Delta(M, \alpha, L)).$$

10.6 Theorem Let (L, u) and (M, v) be D^Δ -pairs and let $\Delta(Y, \alpha, X) \leq M \times L$ be a (v, u) -invariant diagonal subgroup. Then X is u -invariant and Y is v -invariant. Let (X, u') and (Y, v') denote the resulting D^Δ -subpairs of (L, u) and (M, v) , see Remark 2.5(a). Then α is an isomorphism between the D^Δ -pairs (X, u') and (Y, v') . Moreover, one has

$$\begin{aligned} \Psi_{L, u}^{M, v}(\Delta(Y, \alpha, X)) &= \Psi_{Y, v'}^{M, v}(\Delta(Y)) \circ \Psi_{X, u'}^{Y, v'}(\Delta(Y, \alpha, X)) \circ \Psi_{L, u}^{X, u'}(\Delta(X)) \\ &= \text{Ind}_{Y, v'}^{M, v} \circ \text{Iso}(\alpha) \circ \text{Res}_{X, u'}^{L, u}. \end{aligned}$$

Proof The first statements are obvious and the last statement follows immediately from Theorem 10.3. Indeed,

$$\begin{aligned} \left(\Psi_{Y, v'}^{M, v}(\Delta(Y)) \circ \Psi_{X, u'}^{Y, v'}(\Delta(Y, \alpha, X)) \right) \circ \Psi_{L, u}^{X, u'}(\Delta(X)) &= \Psi_{X, u'}^{M, v}(\Delta(Y, \alpha, X)) \circ \Psi_{L, u}^{X, u'}(\Delta(X)) \\ &= \Psi_{L, u}^{M, v}(\Delta(Y, \alpha, X)). \end{aligned}$$

$\square$

10.7 Corollary The category $\mathbb{F}\mathcal{D}^\Delta$ is generated as a preadditive category by the morphisms Ind , Iso and Res .

Proof This follows from Theorems 9.6 and 10.6. $\square$

11 The main theorem

We keep the hypothesis on R from the beginning of Section 8.

For D^Δ -pairs (L, u) and (M, v) and an $((M, v), (L, u))$ -biset Ω , which we view as a left $(M \times L)\langle(v, u)\rangle$ -set, we define ${}^{(v,1)}\Omega$ as the $(M \times L)\langle(v, u)\rangle$ -biset with the same underlying set Ω and an element $a \in (M \times L)\langle(v, u)\rangle$ now acting as $a^{(v,1)}$ acted before. This defines a category automorphism of ${}_{(M,v)}\mathbf{set}_{(L,u)}$. If $\Omega = (M \times L)/Q$ is transitive with (v, u) -invariant $Q \leq M \times L$ (see Proposition 4.4) then ${}^{(v,1)}\Omega \cong (M \times L)/{}^{(v,1)}Q$.

11.1 Lemma *Let (L, u) , (M, v) and (N, w) be D^Δ -pairs. Let*

$$\Gamma = (N \times M)/\Delta(T, \beta, Z)$$

be a transitive diagonal $((N, w), (M, v))$ -biset, and let

$$\Omega = (M \times L)/\Delta(Y, \alpha, X)$$

be a transitive diagonal $((M, v), (L, u))$ -biset. Assume that $\Delta(T, \beta, Z)$ is (w, v) -invariant and that $\Delta(Y, \alpha, X)$ is (v, u) -invariant. Then

$$\Gamma \times_{M,v} {}^{(v,1)}\Omega \cong {}^{(w,1)}(\Gamma \times_{M,v} \Omega)$$

and

$${}^{(w,1)}\Gamma \times_{M,v} \Omega \cong {}^{(w,1)}(\Gamma \times_{M,v} \Omega).$$

Proof We have

$${}^{(v,1)}\Omega \cong (M \times L)/\Delta(Y, v\alpha, X).$$

By Proposition 6.9,

$$\Gamma \times_{M,v} {}^{(v,1)}\Omega \cong \coprod_{m \in Z^v \setminus M^v/Y^v} (N \times L)/\Delta_m,$$

where

$$\Delta_m = \Delta(\beta(Z \cap {}^mY), \beta i_m v \alpha, \alpha^{-1} v^{-1}(Z^m \cap Y)).$$

Since $m \in M^v$, we have $i_m \circ v = v \circ i_m$. Also, the groups Z and Y are v -invariant. So,

$$v^{-1}(Z^m \cap Y) = Z^m \cap Y$$

and

$$\beta i_m v \alpha = \beta v i_m \alpha = w \beta i_m \alpha,$$

since $w\beta = \beta v$. Moreover, since $Z \cap {}^mY$ is v -invariant and $w\beta = \beta v$, the subgroup

$$\beta(Z \cap {}^mY)$$

is w -invariant. Therefore,

$$\Delta_m = \Delta(\beta(Z \cap {}^m Y), \beta i_m v \alpha, \alpha^{-1} v^{-1} (Z^m \cap Y)) = {}^{(w,1)}\Delta(\beta(Z \cap {}^m Y), \beta i_m \alpha, \alpha^{-1} (Z^m \cap Y)).$$

This proves the first isomorphism. The second one is proved similarly. $\square$

11.2 Notation For D^Δ -pairs (M, v) and (L, u) , let

$$\mathcal{I}^\Delta((M, v), (L, u))$$

be the R -submodule of $RB^\Delta((M, v), (L, u))$ generated by the elements

$$[\Omega] - [{}^{(v,1)}\Omega],$$

where Ω runs over the set of transitive diagonal $((M, v), (L, u))$ -bisets. By Lemma 11.1, these subspaces form an ideal in the category $\mathbb{F}\mathcal{P}^\Delta$.

11.3 Definition The quotient category $\overline{R\mathcal{P}^\Delta}$ is defined as follows. Its objects are the D^Δ -pairs, and for two D^Δ -pairs (L, u) and (M, v) ,

$$\mathrm{Hom}_{\overline{R\mathcal{P}^\Delta}}((L, u), (M, v)) = \overline{RB^\Delta}((M, v), (L, u)) := RB^\Delta((M, v), (L, u)) / \mathcal{I}^\Delta((M, v), (L, u)).$$

The composition is induced by the composition in $\mathbb{F}\mathcal{P}^\Delta$.

Note that $\overline{RB^\Delta}((M, v), (L, u))$ is equal to the $\langle(v, 1)\rangle$ -cofixed points of $RB^\Delta((M, v), (L, u))$ under the action of $\langle(v, 1)\rangle$ induced by $\Omega \mapsto {}^{(v,1)}\Omega$. Note that this is a permutation action with $(v, 1)$ -stable basis given by the transitive $((M, v), (L, u))$ -bisets.

For an element $\Omega \in RB^\Delta((M, v), (L, u))$, we write $\overline{\Omega}$ to denote its image in $\overline{RB^\Delta}((M, v), (L, u))$.

11.4 Lemma *The set of elements*

$$\overline{(M \times L)/Q},$$

where Q runs over a set of representatives of $M^v \langle v \rangle \times L^u \langle u \rangle$ -conjugacy classes of (v, u) -invariant diagonal subgroups of $M \times L$, is an R -basis of

$$\overline{RB^\Delta}((M, v), (L, u)).$$

Proof Let $\mathcal{S}$ be the set of (v, u) -invariant diagonal subgroups of $M \times L$. Let $\mathcal{T}$ denote the standard basis of $RB^\Delta((M, v), (L, u))$ consisting of isomorphism classes of transitive $((M, v), (L, u))$ -bisets. By Proposition 6.3, the map $Q \mapsto (M \times L)/Q$ induces a bijection $(M^v \times L * u) \backslash \mathcal{S} \rightarrow \mathcal{T}$. This bijection is $\langle(v, 1)\rangle$ -equivariant, since ${}^{(v,1)}((M \times L)/Q) \cong (M \times L)/{}^{(v,1)}Q$. Under the $\langle(v, 1)\rangle$ -cofixed point construction, the $\langle(v, 1)\rangle$ -orbits of $\mathcal{T}$ parametrize a basis of $\overline{RB^\Delta}((M, v), (L, u))$. By the above, they are in bijection with the $\langle(v, 1)\rangle$ -orbits on $(M^v \times L^u) \backslash \mathcal{S}$. Since every element in $\mathcal{S}$ is already (v, u) -fixed, these are in bijection with the $(M^v \langle v \rangle \times L^u \langle u \rangle)$ -orbits of $\mathcal{S}$. $\square$

11.5 Proposition *Let*

$$\Gamma = (N \times M)/\Delta(T, \beta, Z)$$

and

$$\Omega = (M \times L)/\Delta(Y, \alpha, X)$$

be as in Lemma 11.1. Then in the quotient category $\overline{R\mathcal{P}^\Delta}$, we have

$$\overline{\Gamma} \circ \overline{\Omega} = \sum_{m \in Z^v \setminus M^v/Y^v} \overline{(N \times L)/\Delta_m},$$

where

$$\Delta_m = \Delta(\beta(Z \cap {}^m Y), \beta i_m \alpha, \alpha^{-1}(Z^m \cap Y)).$$

Proof The composition in $\overline{R\mathcal{P}^\Delta}$ is induced by the composition in $R\mathcal{P}^\Delta$. Hence

$$\overline{\Gamma} \circ \overline{\Omega} = \overline{\Gamma \times_{M,v} \Omega}.$$

The result follows now from Proposition 6.9. □

11.6 Theorem *The R -linear categories $\overline{R\mathcal{P}^\Delta}$ and $R\mathcal{D}^\Delta$ are isomorphic.*

Proof Given D^Δ -pairs (M, v) and (L, u) , consider the map

$$\begin{aligned} \phi_{L,u}^{M,v} : \overline{RB^\Delta}((M, v), (L, u)) &\longrightarrow E_{M,v} RT^\Delta(M\langle v \rangle, L\langle u \rangle) E_{L,u} \\ \overline{(M \times L)/Q} &\longmapsto \Psi_{L,u}^{M,v}(Q) \end{aligned}$$

where Q runs over the representatives appearing in Lemma 11.4. By Lemma 11.4 and Theorem 9.6, each $\phi_{L,u}^{M,v}$ is an isomorphism. By Proposition 11.5 and Theorem 10.3, they induce an isomorphism

$$\phi : \overline{R\mathcal{P}^\Delta} \rightarrow R\mathcal{D}^\Delta$$

of R -linear categories. □

11.7 Corollary *The category $\mathcal{F}_{Rpp_k}^\Delta$ of diagonal p -permutation functors over R is equivalent to the category of R -linear functors from $\overline{R\mathcal{P}^\Delta}$ to ${}_R\text{Mod}$.*

Proof This follows from Corollary 8.6 and Theorem 11.6. □

Acknowledgment The first and second authors wish to express their gratitude for the hospitality of the Bilkent Mathematics Department during a visit in summer 2025.

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